Ghent University
Faculty of Science
Department of Mathematics: Algebra and Geometry

Non-associative algebras for exceptional groups

Jari Desmet

Academic Year 2020–2021

Promotor: Prof. Dr. Tom De Medts

Masterproef ingediend tot het behalen van de academische graad
van master in de wiskunde

Preface

First of all, I want to thank Tom De Medts, my supervisor, for guiding me through the perilous adventure that is writing a master's thesis. His many comments and helpful answers to my questions have helped not only this project become better, but have also improved my proofwriting and reasoning skills. Not only that, I want to thank him for his many interesting courses, which steered me into this area of research.

During and outside the many meetings we had over the year, my fellow students Mathias Stout and Michiel Smet were always there for useful discussions around our respective projects, and for that I want to thank them too.

I also want to thank Maurice Chayet and Skip Garibaldi for being very helpful, and swiftly replying to my emails when I had questions about their articles.

There is no way I could have gotten this far without the support of my parents. Thanks to them, I can spend my time doing mathematics, without having to worry about other things.

Lastly, I want to thank Jens Bossaert, for providing the LaTeX-template, and my good friend Francisco Mesquita, for providing me with a Eurodance playlist to make the process of writing more enjoyable.

31 Mei 2021

Contents

Introduction

1 Motivation and context

A common subject that permeates the entirety of mathematics is the study of symmetry. The difficulty of studying symmetry has two main components:

1. What kind of symmetries are possible?

2. Given a symmetry group, what objects have as symmetries that symmetry group (or, are symmetrical in that way)?

The first question is studied in the area of group theory, and the second question can be (at least in part) answered by representation theory.

As an analyst, Sophus Lie was interested in symmetry groups that have *continuous* properties. He wanted to study symmetry elements, that when perturbed by a small amount, give rise to another symmetry element that is similar to the original one. The specific groups he studied are called Lie groups. One of the easiest examples is to look at the symmetry group of a 3-dimensional sphere. This group consists of rotations and reflections. A small change in the rotation axis, or the angle of the rotation gives us a new rotation of the sphere. The same holds for reflections, we can slightly tilt the reflection plane, and the resulting reflection will not be too different from the original one.

The algebraic approach to continuity is to consider zero sets of polynomials. Over the real or complex numbers, polynomials always give continuous images, in the usual sense. By somehow defining symmetries as zero sets of polynomials, this gives rise to the notion of algebraic groups. As an example, in the case of the sphere, the polynomials we consider are simply given by $AA^\top = I_3$, where A is a 3-dimensional matrix.

This thesis is about the representation theory of these algebraic groups. By studying interesting representations of these groups, we aim to gain deeper insight into their structure.

2 Outline

In essence, this thesis is a continuation of the papers [DMVC21] by Tom De Medts & Michiel Van Couwenberghe and [CG21] by Maurice Chayet & Skip Garibaldi. It is important to remark however that we only work over fields of characteristic zero. However, most results should hold under the restrictions put in place by Chayet and Garibaldi. We choose to not treat the technicalities of positive characteristic due to unfamiliarity with the positive characteristic case.

The first two chapters were written with a graduate student in mind, and relay the results of the paper this thesis is based on.

In the first chapter, we introduces all the preliminary notions necessary to understand the results in [CG21]. Most of the definitions and results in this chapter can be found in standard textbooks on root systems, Lie algebras and their representation theory, and we refer to [Hum78] for a comprehensive introduction. The only exceptions to this rule are Section 1.3 and 1.6, where we derive some explicit formulas for certain operators. Anyone familiar with the theory is free to skip this chapter, and go back to it when needed.

The construction of the algebras $A(\mathfrak{g})$ from [CG21] is given in the second chapter. The construction is given in detail, and the main results of [CG21] about these algebras are stated. These algebras were constructed as representations for algebraic groups, mostly with the exceptional types G_2, F_4, E_6, E_7 and in particular E_8 in mind. We also mention some of the structural results Chayet & Garibaldi proved about the algebras in their article.

After these two chapters, the thesis diverges from and expands on [CG21]. In this part of the dissertation, we take a closer look at the algebra of type G_2. Each of the last three chapters has a "main result".

The third chapter is dedicated to finding an alternate description of the algebra of type G_2, denoted $A(\mathfrak{g}_2)$, in terms of the *octonion algebras*. These are, in a way, a very natural generalisation of the complex numbers $\mathbb{C}$ (and the quaternions $\mathbb{H}$). It is known that the group of type G_2 is the automorphism group of such an octonion algebra. We eventually get the following:

Theorem (Main result 1). *The algebra $A(\mathfrak{g}_2)$ is isomorphic to the symmetric square of the* purely imaginary *octonions* $\mathrm{S}^2 W$*, with multiplication given by*

$$\begin{aligned} ab \star cd = &\frac{1}{12}\left(\langle a,c\rangle bd + \langle a,d\rangle bc + \langle b,c\rangle ad + \langle b,d\rangle ac + \langle a,b\rangle cd + \langle c,d\rangle ab\right) \\ &- \frac{1}{48}\left((a*c)(b*d) + (a*d)(b*c)\right). \end{aligned}$$

In this theorem, $*$ denotes the so-called *Maltsev product*, an anticommutative product on the imaginary octonions derived from the usual octonion product.

We handle one specific case of Galois descent on these algebras in the fourth chapter, using the description found in the third chapter. Though this is more of an example than a theorem, we have the following result:

Theorem (Main result 2). *The compact real form of the algebra $A(\mathfrak{g}_2)$ is given by the symmetric square of the compact real form of the octonions, with multiplication as in Main result 1.*

In the fifth and final chapter, we extend one of the results from [CG21]. In this article, they proved that in certain cases (namely when the Dynkin type is F_4 respectively E_8), the automorphism group is "minimal", i.e. it is precisely equal to the original group of type F_4 respectively E_8 (modulo its finite center, since that information is not given by the Lie algebra). We include G_2 into this list.

Theorem (Main result 3). *The automorphism group of $A(\mathfrak{g}_2)$ is precisely the adjoint group of type G_2.*

1 Preliminaries

This chapter introduces most of the preliminaries to this work. Though this is not meant as an introduction to the representation theory of Lie algebras (for excellent introductions, see [Hum78, FH91]), it should contain enough information to understand the following chapters.

1.1 Root systems

Root systems are combinatorial objects by nature. They are beautiful (and useful) because they are extremely symmetrical by definition. These root systems lie at the heart of the classification of simple finite dimensional Lie algebras and by extension, simple algebraic groups.

Definition 1.1.1 (Reflection). Let $n \in \mathbb{N}$ be a natural number. Let $v \in \mathbb{R}^n$ be a non-zero vector, and denote by $\langle \cdot, \cdot \rangle$ a given inner product on $\mathbb{R}^n$. We call the linear map

$$\begin{aligned} \sigma_v \colon \mathbb{R}^n &\to \mathbb{R}^n \colon \\ w &\mapsto w - \frac{2\langle w, v\rangle}{\langle v, v\rangle} v \end{aligned}$$

the *reflection* around v. These reflections are bijective, and preserve the inner product.

We will also denote $\frac{2\langle w,v\rangle}{\langle v,v\rangle}$ by $v(w)$. This notation reflects the linearity in w.

Definition 1.1.2 (Root system). Let n be a natural number. Let $\langle \cdot, \cdot \rangle$ denote a given inner product on $\mathbb{R}^n$. A subset Φ of $\mathbb{R}^n$ is called a root system if it satisfies

(RS1). Φ is finite, spans $\mathbb{R}^n$ and does not contain the zero vector,

(RS2). For every $\alpha \in \Phi$, the only scalar multiples of α in Φ are $\pm\alpha$,

(RS3). For every $\alpha \in \Phi$, the reflection σ_α fixes Φ globally,

(RS4). For every $\alpha, \beta \in \Phi$, we have that $\alpha(\beta) \in \mathbb{Z}$.

We call the elements of Φ roots, and we will call n the rank of the root system Φ.

Since a root system is a subset of a vector space, we can try to choose a basis of the vector space that allows for easy computations.

Definition 1.1.3 (Basis of a root system). A subset Δ of a root system Φ of rank n is called a *basis* of Φ if it satisfies

1. Δ is a basis of $\mathbb{R}^n$,
2. Every root of Φ can be written as $\sum_{\alpha\in\Delta} \lambda_\alpha \alpha$, where either all λ_α are positive integers, or λ_α all are negative integers.

We would of course not define such a basis if it did not exist:

Proposition 1.1.4. *Every root system has a basis.*

Proof. See [Hum78, Theorem 10.1]. ∎

Definition 1.1.5 (Simple reflection). Let Φ be a root system with basis Δ. We will call the elements of Δ simple roots. The corresponding reflections σ_α will be called simple reflections.

Definition 1.1.6. Let Φ be a root system with basis Δ. When a root $\beta = \sum_{\alpha \in \Delta} \lambda_\alpha \alpha$ is such that all λ_α are positive, we call β a positive root. We denote the set of positive roots by Φ^+. Analogously we define the set of negative roots Φ^-.

The group generated by the reflections of the root system is a very transitive group, and it allows us to simplify various arguments.

Definition 1.1.7 (Weyl Group). Let Φ be a root system. The group

$$\mathcal{W} := \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$$

is called the Weyl Group of Φ. Note that $\mathcal{W}$ is generated by the simple reflections.

As in all areas of mathematics, we look for the most fundamental "pieces" of the theory, and then build up from there. For that, we need a notion of irreducibility.

Definition 1.1.8 (Irreducibility of root systems). Let Φ_1, Φ_2 be root systems of rank n_1, n_2 respectively. Embed Φ_1 into $\mathbb{R}^{n_1+n_2}$ by the map $\mathbb{R}^{n_1} \times \{0\} \hookrightarrow \mathbb{R}^{n_1+n_2}$ and call this embedding Φ_1'. Embed Φ_2 into $\mathbb{R}^{n_2}$ by the map $\{0\} \times \mathbb{R}^{n_2} \hookrightarrow \mathbb{R}^{n_1+n_2}$ and call this embedding Φ_2'. Then the set $\Phi_1' \cup \Phi_2'$ is a root system of rank $n_1 + n_2$. We denote it by $\Phi_1 \times \Phi_2$ and call it the direct product of Φ_1 and Φ_2.

A root system Φ is called *irreducible* if it is not isomorphic to the direct product of smaller[1] non-trivial root systems.

Definition 1.1.9 (Height of a root). Let Φ be a root system with basis Δ. For a root $\beta = \sum_{\alpha \in \Delta} \lambda_\alpha \alpha$ we define the height of β with respect to Δ as

$$\mathrm{ht}_\Delta(\beta) := \sum_{\alpha \in \Delta} \lambda_\alpha.$$

We will omit mentioning Δ when it is clear from the context.

In the following chapters, we will mostly work with irreducible root systems. This next proposition is good to have in mind when thinking about irreducible root systems.

Proposition 1.1.10 (Properties of irreducible root systems). *Let Φ be an irreducible root system with basis Δ.*

1. *The root system Φ has at most two different root lengths $\langle \alpha, \alpha \rangle^{\frac{1}{2}}$. We will call the roots with shortest root length the short roots, and the roots with longest root length the long roots.*
2. *The root system Φ has a unique root $\tilde{\alpha}$ with maximal height. The root $\tilde{\alpha}$ is always a long root.*
3. *Among the short roots, there is also a unique short root with maximal height.*
4. *The Weyl Group of Φ acts transitively on the short roots and on the long roots, respectively.*

Proof. See [Hum78, §10.4]. ∎

[1] "smaller" here is determined by the rank of the root system.

The previous proposition leads to another concept we will need in Section 1.3.

Notation 1.1.11. Let Φ be an irreducible root system. Denote by ν the square length ratio of a long root to a short root. For α a root of Φ we define

$$\nu_\alpha := \begin{cases} \nu & \text{if } \alpha \text{ is short,} \\ 1 & \text{if } \alpha \text{ is long.} \end{cases}$$

Equivalently, we have $\nu_\alpha = \frac{\langle \alpha,\alpha \rangle}{\langle \tilde{\alpha},\tilde{\alpha} \rangle}$, where $\tilde{\alpha}$ is the highest root (with respect to a given basis).

In terms of this new parameter associated to a root system, we will be able to describe the integers $\alpha(\beta)$.

Lemma 1.1.12. *Let Φ be an irreducible root system with basis Δ. Let α, β be different roots, and suppose α is long. Then $\alpha(\beta) = 0$ or ± 1.*

Proof. Pick a basis Δ of Φ. Since the Weyl group works transitively on the long roots, and leaves the inner product unchanged, we can assume $\alpha = \tilde{\alpha}$, the highest root.

Suppose β is a positive root first.

The value of $\alpha(\beta)$ cannot be less than 0 since α is the highest root. Otherwise, $\sigma_\alpha(\beta)$ would have a height greater than α. If $\alpha(\beta)$ is greater than 1, then $-\sigma_\alpha(\beta) = \alpha(\beta)\alpha - \beta$ would be a root in Φ with greater height than α, unless $\beta = \alpha$.

The case where β is a negative root is analogous. ∎

To get rid of the assumption that α is long, we will need the concept of a *dual root.*

Definition 1.1.13. For a root α of a root system Φ, denote by $\alpha^\vee := \frac{2\alpha}{\langle \alpha,\alpha \rangle}$ its dual root.

Proposition 1.1.14 (Dual root system). *Let Φ be a root system with basis Δ. Then*

$$\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$$

is also a root system, and

$$\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$$

is a basis for $\Phi^\vee$. Moreover, if Φ is irreducible, then so is $\Phi^\vee$.

Proof. The conditions (RS1) and (RS2) hold for $\Phi^\vee$, since they hold for Φ. We now prove (RS4). For this, we compute $\alpha^\vee(\beta^\vee)$.

$$\begin{aligned} \alpha^\vee(\beta^\vee) &= \frac{\langle \alpha^\vee, \beta^\vee \rangle}{\langle \alpha^\vee, \alpha^\vee \rangle} \\ &= \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \\ &= \beta(\alpha) \in \mathbb{Z}. \end{aligned}$$

Lastly, we check (RS4). More specifically, we prove that $\sigma_{\alpha^\vee}(\beta^\vee) = \sigma_\alpha(\beta)^\vee$.

$$\begin{aligned} \sigma_{\alpha^\vee}(\beta^\vee) &= \beta^\vee - \alpha^\vee(\beta^\vee)\alpha^\vee \\ &= \frac{\beta - \alpha(\beta)\alpha}{\langle \beta, \beta \rangle}. \end{aligned}$$

Note that $\langle \sigma_\alpha(\beta), \sigma_\alpha(\beta) \rangle = \langle \beta, \beta \rangle$, proving (RS4).

Lastly, note that if $\Phi^\vee = \Phi_1 \times \Phi_2$, then $\Phi = (\Phi^\vee)^\vee = \Phi_1^\vee \times \Phi_2^\vee$. In other words, if the dual root system is reducible, then so is the original. ∎

As promised, with the dual root system, we will extend Lemma 1.1.12.

Corollary 1.1.15. *Let α, β be different roots of an irreducible root system Φ. Denote by ν the square ratio of the length of a long root to the length of a short root.*

1. *if α is long, then $\alpha(\beta) = 0$ or ± 1.*
2. *if α, β are both short, then $\alpha(\beta) = 0$ or ± 1.*
3. *if α is short and β is long, then $\alpha(\beta) = 0$ or $\pm \nu$.*

Proof. If α is long, this is precisely Lemma 1.1.12.

When α, β are both short roots, we can simply note that $\alpha(\beta) = \beta^\vee(\alpha^\vee)$ and use Proposition 1.1.14.

When α is short, and β is long, we can use Lemma 1.1.12 by noting that $\alpha(\beta) = \nu\beta(\alpha)$. ■

Remark 1.1.16. If we want to summarise this in one formula, then this corollary becomes:

$$\alpha(\beta) = 0 \text{ or } \pm \left\lceil \frac{\nu_\alpha}{\nu_\beta} \right\rceil .$$

There is one more important concept that we will need, the Dual Coxeter number. This combinatorial number shows up in Proposition 1.3.1, and for that reason, it will also show up in the construction of $A(\mathfrak{g})$.

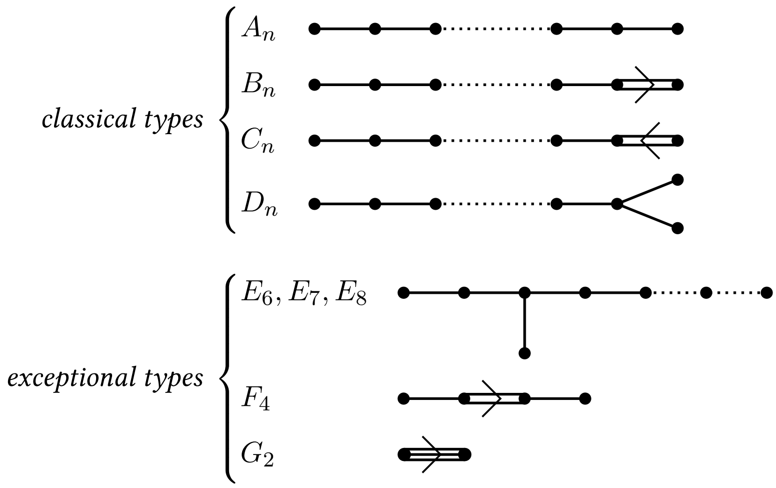
Definition 1.1.17 ((Dual) Coxeter number). Let Φ be an irreducible root system with highest root $\tilde{\alpha}$ (with respect to a basis of simple roots $\{\alpha_1, \dots, \alpha_n\} = \Delta$).

1. Define the *Coxeter number* by
$$h := 1 + \sum_i m_i,$$
where $\tilde{\alpha} = \sum_i m_i \alpha_i$.
2. Define the *Dual Coxeter number* by
$$h^\vee := 1 + \sum_i m_i',$$
where $\tilde{\alpha}^\vee = \sum_i m_i' \alpha_i^\vee$.

We end this section with the following classification result, which is integral to the results of this thesis. First we introduce the Dynkin diagrams.

Definition 1.1.18. Let Φ be a root system with basis Δ. We define the Dynkin diagram to be the graph with vertex set Δ, where $\alpha, \beta \in \Delta$ are adjoined by exactly $\alpha(\beta) \cdot \beta(\alpha)$ edges. When $\alpha(\beta) \cdot \beta(\alpha)$ is greater than 1, we label the edges with an arrow pointing to the smallest root (e.g. if $\alpha(\beta) > \beta(\alpha)$, the arrow points to α).

Theorem 1.1.19. *An irreducible root system is isomorphic to one of the root systems with Dynkin diagram:*



Proof. [Hum78, Theorem 11.4] ■

1.2 Lie algebras

Lie algebras are the most well known class of non-associative algebras. They are the "linearization" of Lie groups, and bring us one step closer to the objects we want to research (that is, algebraic groups).

Definition 1.2.1. 1. A Lie algebra is a vector space $\mathfrak{g}$ over a field k together with a bilinear product $[\cdot, \cdot]$ satisfying

(L1) $[X, X] = 0$,

(L2) (Jacobi identity) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$,

for all $X, Y, Z \in \mathfrak{g}$.

2. For any $X \in \mathfrak{g}$ we define the adjoint of X, denoted $\operatorname{ad} X \in \operatorname{End}(\mathfrak{g})$ by

$$(\operatorname{ad} X)(Y) := [X, Y] \text{ for all } Y \in \mathfrak{g}.$$

3. The Killing form of a Lie algebra $\mathfrak{g}$ is defined to be the map

$$K \colon \mathfrak{g} \times \mathfrak{g} \to k \colon (A, B) \mapsto \operatorname{Tr}(\operatorname{ad} A \circ \operatorname{ad} B).$$

This is a symmetric bilinear operator. When working with more than one Lie algebra, we will index the Killing form by the corresponding algebra to avoid confusion.

4. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a nilpotent subalgebra that is normalised by itself, i.e.

a) there is an $n \in \mathbb{N}$ such that for all $H_i \in \mathfrak{h}$, $[H_1, [H_2, [\ldots, [H_{n-1}, H_n] \ldots]]] = 0$,

b) for any $X \in \mathfrak{g}$, if $[X, H]$ is contained in $\mathfrak{h}$ for all $H \in \mathfrak{h}$, then so is X.

Example 1.2.2. Let A be any associative k-algebra. Then A, equipped with the commutator product

$$[x, y] := xy - yx,$$

for all $x, y \in A$ is a Lie algebra. In fact, this is the prototype of a Lie algebra, and any Lie algebra can be seen to be of this form (see Definition 1.6.1).

We are mostly interested in these Cartan subalgebras, and their interplay with the Killing form. Recall that an algebra is semisimple if it can be written as the direct sum of simple algebras (i.e., algebras with no nontrivial ideals).

Lemma 1.2.3. *For a semisimple Lie algebra $\mathfrak{g}$ over a field k, the Killing form K is nondegenerate on $\mathfrak{g}$, and remains nondegenerate after restricting to a Cartan subalgebra $\mathfrak{h}$.*

Proof. [Hum78, §5.1 & §8.2]. ■

To any semisimple Lie algebra, we can associate a root system.

In a nutshell, the root system of $\mathfrak{g}$ is constructed as follows: we pull back the Killing form to the dual vector space $\mathfrak{h}^*$. We denote by Φ the elements of $\mathfrak{h}^*$ that have a non-zero eigenspace in $\mathfrak{g}$. Then one can prove that Φ together with the pulled back Killing form forms a root system, and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ denotes the 1-dimensional eigenspace of $\alpha \in \Phi$.

As in the previous section, we denote a given basis of Φ by Δ, and the positive roots with respect to Δ by Φ^+.

Notation 1.2.4. Let f be a bilinear form on $\mathfrak{g}$. For $A \in \mathfrak{g}$ we denote the map that sends B to $f(A, B)$ for all $B \in \mathfrak{g}$ by $f(A, \_)$. Analogously, for $A, B \in \mathfrak{g}$ we denote the map that sends C to $Bf(A, C)$ for all $C \in \mathfrak{g}$ by $Bf(A, \_)$.

Lemma 1.2.5 (Some Chevalley properties). *For each $\lambda \in \mathfrak{h}^*$ there is a unique $H'_\lambda \in \mathfrak{h}$ such that $\lambda(H) = K(H'_\lambda, H)$ for all $H \in \mathfrak{h}$.*

Proof. It follows immediately from the nondegeneracy of the Killing form that the linear map

$$\phi\colon \mathfrak{h} \to \mathfrak{h}^*\colon H \mapsto K(H, \_)$$

is injective. As $\mathfrak{h}$ is finite dimensional, this also means it is a bijection by dimension count. ■

Definition 1.2.6. Define the map

$$\begin{aligned} \langle \cdot, \cdot \rangle\colon \mathfrak{h}^* \times \mathfrak{h}^* &\to k \\ (\lambda, \mu) &\mapsto K(H'_\lambda, H'_\mu) \end{aligned}$$

where H'_λ, H'_μ are as in the previous lemma.

Note that $\langle \cdot, \cdot \rangle$ is also a nondegenerate scalar product on $\mathfrak{h}^*$.

Definition 1.2.7. We will call an element X of $\mathfrak{g} \setminus \{0\}$ an eigenvector of weight $\alpha \in \mathfrak{h}^*$ if for any $H \in \mathfrak{h}$, the equation $[H, X] = \alpha(H)X$ holds. We will call $\alpha \in \mathfrak{h}^*$ a weight if there is an eigenvector of weight α.

Proposition 1.2.8. *The set of weights Φ forms a root system with respect to $\langle \cdot, \cdot \rangle$.*

Proof. See [Hum78, Theorem 8.5]. ■

Now we just need one more lemma to be able to define the Chevalley basis of a Lie algebra.

Proposition 1.2.9. *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field k with $\operatorname{char} k = 0$, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{g}_\alpha$ the space of eigenvectors of weight α. The following holds:*

1. $[\mathfrak{h}, \mathfrak{h}] = 0$,
2. $\dim \mathfrak{g}_\alpha = 1$ *for any nonzero weight* α,
3. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ *for* $\alpha \neq -\beta$,
4. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = kH'_\alpha$ *for* $\alpha \in \Phi$.

Proof. [Hum78, Proposition 8.1,Proposition 8.2, Proposition 8.3 and Corollary 15.3] ■

Definition 1.2.10. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\alpha \in \Phi$, define $H_\alpha = \frac{2H'_\alpha}{\langle \alpha, \alpha \rangle}$. We call it the coroot of α.

For the proof of this theorem we refer to [Car89], as [Hum78] does not state this theorem in full.

Theorem 1.2.11 (Chevalley). *Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field k of characteristic 0 and let*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be a Cartan decomposition. Let H_α be the coroot of α. Then for each root $\alpha \in \Phi$, we can choose an element $X_\alpha \in \mathfrak{g}_\alpha$ such that

$$\begin{aligned} [X_\alpha, X_{-\alpha}] &= H_\alpha, \\ [H_\alpha, X_\beta] &= \alpha(\beta) X_\beta, \\ [X_\alpha, X_\gamma] &= \pm(p+1) X_{\alpha+\gamma}, \end{aligned}$$

where p is the greatest integer for which $\gamma - p\alpha \in \Phi$, for all $\alpha, \beta, \gamma \in \Phi$, and $\gamma \neq -\alpha$. Moroever, by this multiplication rule, for any irreducible root system, there exists a simple Lie algebra associated to that root system.

Proof. [Car89, Theorem 4.2.1] ■

Definition 1.2.12 (Chevalley basis). We call a basis of $\mathfrak{g}$ satisfying the properties of the previous theorem a Chevalley basis of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and we call a Lie algebra *split* if it contains a Chevalley basis. Any Lie algebra over an algebraically closed field of characteristic 0 is split.

Whenever we have such a basis, we can translate the symmetries of the associated root systems into automorphisms of the Lie algebra.

Proposition 1.2.13. *Let $\mathfrak{g}$ be a Lie algebra with associated root system Φ and Chevalley basis $\mathcal{C}$. For any $\alpha \in \Phi$, the map*

$$\begin{aligned} \sigma_\alpha \colon \mathfrak{g} &\to \mathfrak{g} \\ e_\beta &\mapsto e_{\sigma_\alpha(\beta)} \\ h_\beta &\mapsto h_{\sigma_\alpha(\beta)} \end{aligned}$$

(and linearly extending this definition) is an automorphism of $\mathfrak{g}$.

Proof. By definition, it is a linear bijection, since it maps the basis $\mathcal{C}$ to itself. As the reflections of the root system Φ preserve the inner product, the map σ_α preserves the Lie brackets, by Theorem 1.2.11. ∎

Definition 1.2.14. Let $\mathfrak{g}$ be a Lie algebra with associated root system Φ. Then

$$\mathcal{W} := \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$$

is called the *Weyl group* of the Lie algebra $\mathfrak{g}$, with respect to the Chevalley basis $\mathcal{C}$.

1.3 The Killing form in terms of the root system

In this section, we use the material in the seminar "Conjugacy Classes" by Springer & Steinberg [SS70] to get some nice formulas for the Killing form with respect to a Chevalley basis $\mathcal{C}$.

Proposition 1.3.1. *Let $\mathfrak{g}$ be a simple Lie algebra with Chevalley basis $\{X_\alpha \mid \alpha \in \Phi\} \cup \{H_\alpha \mid \alpha \in \Delta\}$. Then, for $\alpha, \beta \in \Phi$, we have*

$$K(H_\alpha, H_\beta) = 2\alpha(\beta)\nu_\beta h^\vee.$$

Proof. Denote $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Recall that the scalar product on the roots is defined (Definition 1.2.6) as

$$\begin{aligned}\langle \alpha, \beta \rangle &= K(H'_\alpha, H'_\beta) \\ &= K\left(\frac{\langle \alpha, \alpha \rangle}{2} H_\alpha, \frac{\langle \beta, \beta \rangle}{2} H_\beta\right).\end{aligned}$$

So we already have that

$$K(H_\alpha, H_\beta) = 2\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\frac{1}{\langle \beta, \beta \rangle}.$$

It remains to show that $\frac{1}{\langle \beta, \beta \rangle} = \nu_\beta h^\vee$.

We can rewrite this completely in terms of the root system by applying the definition of $\langle \beta, \beta \rangle$:

$$\begin{aligned}\langle \beta, \beta \rangle &= \operatorname{Tr}(\operatorname{ad} H'_\beta \circ \operatorname{ad} H'_\beta) \\ &= \sum_{\alpha \in \Phi} \langle \alpha, \beta \rangle^2,\end{aligned}$$

since $(\operatorname{ad} H'_\beta \circ \operatorname{ad} H'_\beta)(X_\alpha) = \langle \alpha, \beta \rangle^2 X_\alpha$ for all $\alpha \in \Phi$, and $(\operatorname{ad} H'_\beta \circ \operatorname{ad} H'_\beta)(H) = 0$ for all $H \in \mathfrak{h}$ hold.

We now divide both sides by $\langle \beta, \beta \rangle^2$ to get

$$\frac{1}{\langle \beta, \beta \rangle} = \frac{1}{4}\sum_{\alpha \in \Phi} \langle \alpha, \beta^\vee \rangle^2.$$

1. Suppose first that β is long, i.e. $\nu_\beta = 1$.

   First note that for any $i = 1, \dots, n$, a simple reflection σ_{α_i} permutes the positive roots other than α_i, and sends α_i to $-\alpha_i$ [Hum78, Lemma 10.2.B]. So if $\alpha \in \Phi^+$ is not equal to α_i, we have $\sigma_{\alpha_i}(\alpha) \in \Phi^+$, and $\langle \alpha, \alpha_i^\vee \rangle = -\langle \sigma_{\alpha_i}(\alpha), \alpha_i^\vee \rangle$. We obtain

   $$\sum_{\alpha \in \Phi^+} \langle \alpha, \alpha_i^\vee \rangle = \langle \alpha_i, \alpha_i^\vee \rangle = 2.$$

   Combining this with the equation $\tilde{\alpha}^\vee = \sum_i m_i \alpha_i^\vee$ we get

   $$\sum_{\alpha \in \Phi^+} \langle \alpha, \tilde{\alpha}^\vee \rangle = 2(h^\vee - 1).$$

   Now, as $\tilde{\alpha}$ is the highest root, $\langle \alpha, \tilde{\alpha}^\vee \rangle = \tilde{\alpha}(\alpha) = 0$ or 1 if $\alpha \neq \tilde{\alpha}$ by Corollary 1.1.15.

   In other words, squaring each term in the left-hand side of the equation only adds 2, from squaring the term $\langle \tilde{\alpha}, \tilde{\alpha}^\vee \rangle$. Finally we get

   $$\sum_{\alpha \in \Phi^+} \langle \alpha, \tilde{\alpha}^\vee \rangle^2 = 2h^\vee.$$

   Doubling this equation by also adding Φ^- into the summation, we get what we wanted to prove for $\beta = \tilde{\alpha}$. As the Weyl group acts transitively on the set of long roots, and permutes the roots of Φ, we get what we wanted for any long root β.

2. Now, if β is a short root, the formula is clear, since $\langle \beta, \beta \rangle = \frac{1}{\nu_\beta} \langle \tilde{\alpha}, \tilde{\alpha} \rangle$. ■

Corollary 1.3.2. *For $\alpha \in \Phi$, and $C \in \mathcal{C}$, we have $K(X_\alpha, C) = 0$ unless $C = X_{-\alpha}$, and then we have*

$$K(X_\alpha, X_{-\alpha}) = 2\nu_\alpha h^\vee.$$

Proof. For an element $C \in \mathcal{C}$ of weight β, the map $(\operatorname{ad} X_\alpha \circ \operatorname{ad} C)$ sends a weight vector of weight γ to a weight vector of weight $\alpha + \beta + \gamma$. It is then easy to see that for $(\operatorname{ad} X_\alpha \circ \operatorname{ad} C)$ to have nonzero trace, we need that $\beta = -\alpha$.

The second part follows by the $\mathfrak{g}$-invariance of the Killing form. Indeed, we have

$$\begin{aligned} & K([X_\alpha, H_\alpha], X_{-\alpha}) + K(H_\alpha, [X_\alpha, X_{-\alpha}]) = 0 \\ \iff & K(X_\alpha, X_{-\alpha}) = \frac{1}{2} K(H_\alpha, H_\alpha). \end{aligned}$$ ■

We end this section with explicit formulas for inner products in the root system. Notice the dual Coxeter number showing up.

Corollary 1.3.3. *For any root $\alpha \in \Phi$, we have that*

$$\langle \alpha, \alpha \rangle = \frac{1}{\nu_\alpha h^\vee}.$$

Lemma 1.3.4. *For $\tilde{\alpha}$ the highest root of Φ and δ half the sum of the positive roots, we have*

1. $\langle \tilde{\alpha}, \delta \rangle = \frac{1}{2} - \frac{1}{2h^\vee}$,
2. $\langle \tilde{\alpha}, \tilde{\alpha} + 2\delta \rangle = 1$.

Proof. (From [Hum78, Section 13]) The fundamental dominant weights λ_j are defined to be the dual basis of $\{\alpha_i^\vee \mid \alpha_i \in \Delta\}$ [Hum78, §13.1, p.67]. Now, it turns out that we can write $\delta = \sum_j \lambda_j$ [Hum78, Lemma 13.3.A, p.70]. Using this, we get

$$\begin{aligned} \langle \tilde{\alpha}, \delta \rangle &= \frac{1}{2} \langle \tilde{\alpha}, \tilde{\alpha} \rangle \cdot \left\langle \sum_i m_i \alpha_i^\vee, \sum_j \lambda_j \right\rangle \\ &= \frac{1}{2h^\vee} \sum_i m_i \\ &= \frac{1}{2} - \frac{1}{2h^\vee}, \end{aligned}$$

which proves 1. Now 2 follows from 1, using the fact that $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = \frac{1}{h^\vee}$. ■

1.4 Representation theory of Lie algebras

The main objects of interest in this thesis are representations of Lie algebras (actually, of their associated algebraic groups). To construct the new algebra $A(\mathfrak{g})$, we will need some theoretical concepts, and a few elementary examples. We will introduce them here.

Definition 1.4.1 (Representation of a Lie algebra). Let $\mathfrak{g}$ be a Lie algebra. Let V be a vector space equipped with a morphism of Lie algebras

$$\rho\colon \mathfrak{g} \to \mathrm{End}(V).$$

Then we call (V, ρ) a *representation* of $\mathfrak{g}$. Equivalently, we define a $\mathfrak{g}$-module to be a vector space V together with a module multiplication $\cdot\colon \mathfrak{g} \times V \to V$ satisfying

1. $\cdot$ is linear in both arguments,
2. $X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = [X, Y] \cdot v$ for all $X, Y \in \mathfrak{g}$ and $v \in V$.

Example 1.4.2.

1. The field k is a trivial $\mathfrak{g}$-module, with multiplication given by

$$X \cdot a = 0$$

for all $X \in \mathfrak{g}$ and $a \in k$.

2. For any Lie algebra $\mathfrak{g}$, the underlying vector space, together with the map

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ X &\mapsto \mathrm{ad}\, X \end{aligned}$$

forms a representation of $\mathfrak{g}$. Indeed, $\mathrm{ad}[X, Y] = [\mathrm{ad}\, X, \mathrm{ad}\, Y]$ for all $X, Y \in \mathfrak{g}$, by the Jacobi identity.

3. For any Lie algebra $\mathfrak{g}$, the vector space $\mathrm{End}(\mathfrak{g})$ is also a $\mathfrak{g}$-module, by considering the multiplication

$$X \cdot \phi = [\mathrm{ad}\, X, \phi] \text{ for all } X \in \mathfrak{g}, \phi \in \mathrm{End}(\mathfrak{g}),$$

where in the right-hand side, the brackets denote the usual commutator of operators.

This is a module, since

$$\begin{aligned}[X,Y]\cdot(\phi) &= [\mathrm{ad}([X,Y]),\phi] \\ &= -[\phi,[\mathrm{ad}\,X,\mathrm{ad}\,Y]] \\ &= [\mathrm{ad}\,X,[\mathrm{ad}\,Y,\phi]] - [\mathrm{ad}\,Y,[\mathrm{ad}\,X,\phi]] \\ &= X\cdot(Y\cdot\phi) - Y\cdot(X\cdot\phi).\end{aligned}$$

The last example we need uses tensor products.

Definition 1.4.3 (Symmetric square). For any vector space V, we define the *symmetric square* of V as

$$S^2V := {}^{V\otimes V}\!\big/_{(v\otimes w - w\otimes v \mid v,w\in V)}.$$

Example 1.4.4. The vector space $\mathrm{S}^2\,\mathfrak{g}$ can also be equipped with a $\mathfrak{g}$ representation, by the morphism

$$\begin{aligned}\Phi\colon \mathfrak{g} &\to \mathrm{End}(\mathrm{S}^2\,\mathfrak{g}) \\ X &\mapsto (YZ \mapsto [X,Y]Z + Y[X,Z]).\end{aligned}$$

We check this is a morphism of Lie algebras:

$$\begin{aligned}\Phi([X_1,X_2])(YZ) &= [[X_1,X_2],Y]Z + Y[[X_1,X_2],Z] \\ &= [X_1,[X_2,Y]]Z - [X_2,[X_1,Y]]Z \\ &\quad + Y[X_1,[X_2,Z]] - Y[X_2,[X_1,Z] \\ &= [X_1,[X_2,Y]]Z + Y[X_1,[X_2,Z]] + [X_1,Y][X_2,Z] + [X_2,Y][X_1,Z] \\ &\quad - [X_2,[X_1,Y]]Z - Y[X_2,[X_1,Z]] - [X_1,Y][X_2,Z] - [X_2,Y][X_1,Z] \\ &= [\Phi(X_1),\Phi(X_2)](YZ).\end{aligned}$$

In any representation theory, we usually have (some variant of) Schur's Lemma.

Lemma 1.4.5 (Schur's Lemma). *Suppose $\mathfrak{g}$ is a Lie algebra over an algebraically closed field k that acts irreducibly on V, and $\varphi \in \mathrm{End}(V)$ is an endomorphism commuting with $\mathfrak{g}$, then φ is just an endomorphism multiplying every vector by the same scalar.*

Proof. Since we work over an algebraically closed field, we can find an eigenvector $v \in V$ of φ with eigenvalue $a \in k$. Then Xv is also an eigenvector with eigenvalue a for all X in $\mathfrak{g}$, since

$$\phi Xv = X\phi v = aXv$$

holds. But since $\mathfrak{g}$ acts irreducibly on V, the space $\mathfrak{g}\cdot v$ is equal to V. This means ϕ acts as a scalar a on V. ■

In case k is algebraically closed of characteristic 0, and $\mathfrak{g}$ is a semisimple Lie algebra, the representation theory is well known. The following structure results are important for the rest of this thesis.

Theorem 1.4.6 (Weyl's Theorem). *Any finite dimensional representation of a semisimple Lie algebra is completely reducible (i.e. decomposes into the direct sum of irreducible components).*

Proof. See [Hum78, Theorem 6.3]. ■

Definition 1.4.7. Let Φ be a root system of rank n. We call $\lambda \in \mathbb{R}^n$ a weight of Φ if

$$2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

for all $\alpha \in \Phi$. We denote the set of weights by Λ. We call a weight $\lambda \in \Lambda$ dominant (with respect to a given basis Δ) if

$$2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \geq 0$$

for all $\alpha \in \Delta$. We denote the set of dominant weights by Λ^+.

Theorem 1.4.8. *Let $\mathfrak{g}$ be a semisimple Lie algebra with associated root system Φ, basis Δ and Chevalley basis C. Then for every dominant weight $\lambda \in \Lambda^+$ of the root system Φ there is a unique finite-dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ that is generated (as a module) by a vector v^+ satisfying*

1. *$X_\alpha \cdot v^+ = 0$ for every positive root α,*
2. *$H \cdot v^+ = \lambda(H)v^+$ for all $H \in \mathfrak{h}$.*

The vector v^+ is called the maximal vector. Moreover, any finite-dimensional irreducible module V is of the form $V = V(\lambda)$ for a certain dominant weight $\lambda \in \Lambda^+$.

Proof. [Hum78, Corollary 21.2] ∎

Remark 1.4.9. Over positive characteristic there are modules denoted $V(\lambda)$ called the *Weyl Modules*. These are in general not irreducible, though they are over characteristic 0. Over positive characteristic we can still classify the irreducible modules using weights, but then they will be denoted $L(\lambda)$ (See [Jan03]). The only place we need this notation is in Proposition 2.6.1, to ensure that the statement of the proposition is actually correct.

Remark 1.4.10. Usually, when we talk about specific roots or weights for a certain simple Lie algebra, we label them in a way known as *Bourbaki labelling*, as is common in the literature. This labelling is given in [Bou02, Plates I-IX,p.264-290].

The last thing we need about these representations are some formulas about their characters.

Proposition 1.4.11 (Weyl). *If λ is a dominant weight, and δ half the sum of the positive roots, then the dimension of $V(\lambda)$ is given by*

$$\dim V(\lambda) = \frac{\Pi_{\alpha \in \Phi^+} \alpha(\lambda + \delta)}{\Pi_{\alpha \in \Phi^+} \alpha(\delta)}.$$

Proposition 1.4.12 (Steinberg). *Let $n(\lambda)$ be the number of times $V(\lambda)$ occurs in $V(\lambda') \otimes V(\lambda'')$, with all three weights dominant. Denote by δ half the sum of the positive roots. Then $n(\lambda)$ is given by*

$$n(\lambda) = \sum_{\sigma \in \mathcal{W}} \sum_{\tau \in \mathcal{W}} \operatorname{sgn}(\sigma\tau) p(\lambda + 2\delta - \sigma(\lambda' + \delta) - \tau(\lambda'' + \delta)).$$

Here $p(\mu)$ denotes the number of sets of nonnegative integers $\{\kappa_\alpha, \alpha \in \Phi^+\}$ for which $-\mu = \sum k_\alpha \alpha$, for a weight μ.

The primary reason we want to know these numbers $n(\lambda)$ is because they tell us how many maps there are between representations of Lie algebras.

Proposition 1.4.13. *Suppose E is a $\mathfrak{g}$-representation, and $n(\lambda)$ is the number of times the irreducible representation $V(\lambda)$ occurs in E. Then the vector space of $\mathfrak{g}$-equivariant homomorphisms $E \to V(\lambda)$ has dimension exactly $n(\lambda)$.*

Proof. Let $V_i,\ i = 1, \ldots, n(\lambda)$ be the copies of $V(\lambda)$ in E. Then we can define a $\mathfrak{g}$-equivariant homomorphism η_i for any V_i by sending V_i to V and its complement in E to 0. Then the η_i are linearly independent, so the dimension of the homomorphism space is at least $n(\lambda)$. Now suppose η is any other morphism $E \to V$. Then by Schur's lemma, $\eta|_{V_i}$ has to be a scalar multiple of V, say $\eta|_{V_i} = \mu_i \eta_i$. Suppose W is another irreducible component of E non-isomorphic. Then, because W is irreducible, $\eta|_W$ has to be an isomorphism or the zero map, as its kernel is a submodule of W. Thus $\eta|_W = 0$. We have proven that $\eta = \mu_i \eta_i$. The statement follows. ■

Remark 1.4.14. These and other formulas involving the representation theory of Lie algebras can be computed using a computer algebra package called LiE[2]. Whenever we mention dimensions of certain representations or morphism spaces, these are computed using this package.

1.5 Endomorphisms of Lie algebras

In this section we introduce some basic terminology for vector spaces equipped with a nondegenerate bilinear symmetric form.

We assume $\mathfrak{g}$ is a vector space equipped with a nondegenerate bilinear symmetric form $f\colon \mathfrak{g} \times \mathfrak{g} \to k$.

Lemma 1.5.1. *We have an isomorphism*

$$\begin{aligned} \mathfrak{g} \otimes \mathfrak{g} &\tilde{\to} \operatorname{End}(\mathfrak{g}) \\ A \otimes B &\mapsto Af(B, \_). \end{aligned}$$

Proof. This map is injective by the nondegeneracy of f. It is also surjective by dimension count. ■

Definition 1.5.2 (Transpose operator). Let $T \in \operatorname{End}(\mathfrak{g})$. Then the unique operator $T^\top$ such that

$$f(T(X), Y) = f(X, T^\top(Y))$$

for all $X, Y \in \mathfrak{g}$ is called the *transpose operator*.

If we write $T = \sum_i X_i f(Y_i, \_)$, then $T^\top = \sum_i Y_i f(X_i, \_)$.

Definition 1.5.3 (Symmetric operators). We call an operator $T \in \operatorname{End}(\mathfrak{g})$ if $T = T^\top$. We denote by $\mathcal{H}(\mathfrak{g})$ the vector space of all symmetric operators. When confusion could arise, we index the notation by f, e.g. $\mathcal{H}_f(\mathfrak{g})$.

We can actually regard the space of symmetric operators as a subalgebra of $\operatorname{End}(\mathfrak{g})$, with respect to the right product.

Notation 1.5.4. Suppose we have a k-algebra structure $(V, +, \cdot)$, with k a field, $\operatorname{char} k \neq 2$. Then we define

$$A \bullet B := \frac{A \cdot B + B \cdot A}{2}.$$

We call this product the *Jordan product.*

[2]See `http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/`.

Definition 1.5.5. A *Jordan algebra* over a field k is a commutative k-algebra A satisfying

$$(xy)(xx) = x(y(xx)) \text{ for all } x, y \in A.$$

We now that $\mathrm{End}(\mathfrak{g})$ has a k-algebra structure by composition, thus we also have a Jordan product on $\mathrm{End}(\mathfrak{g})$.

Proposition 1.5.6. *The space* $\mathrm{End}(\mathfrak{g})$ *together with the Jordan product* $\bullet$ *is a Jordan algebra, and the space of symmetric operators* $\mathcal{H}$ *is a Jordan subalgebra of* $\mathrm{End}(\mathfrak{g})$.

Proof. It is clear that $(\mathrm{End}(\mathfrak{g}), \bullet)$ is a commutative k-algebra with unit $\mathrm{id}_{\mathfrak{g}}$. It remains to check the identity for Jordan algebras. Let $f, g \in \mathrm{End}(\mathfrak{g})$. Then we have

$$\begin{aligned}
(f \bullet g) \bullet (f \bullet f) &= \frac{fg + gf}{2} \bullet f^2 \\
&= \frac{fgf^2 + gf^3 + f^3 g + f^2 gf}{4} \\
&= f \bullet \frac{gf^2 + f^2 g}{2} \\
&= f \bullet (g \bullet (f \bullet f)).
\end{aligned}$$

The fact that the symmetric operators form a subalgebra is a consequence of the identity $(AB)^\top = B^\top A^\top$ for all endomorphisms A, B. Then it follows that $(A \bullet B)^\top = A \bullet B$ for symmetric operators A, B. ■

1.6 About Casimir Elements

In this section, we make use of Bourbaki's chapter VIII on Lie Algebras [Bou05], and the definition of the Casimir element in Humphreys' Introduction to Lie Algebras and Representation Theory [Hum78].

The aim is to prove Lemma 2.9 from the paper by Garibaldi & Chayet [CG21] for algebraically closed fields k of characteristic zero.

Definition 1.6.1 (Quadratic Casimir Element).

1. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. This is the unique associative unital k-algebra A with a Lie algebra morphism $i\colon \mathfrak{g} \to A$ such that for every associative algebra B and a Lie algebra morphism $\phi\colon \mathfrak{g} \to B$ there is a unique algebra morphism $\phi'\colon A \to B$ such that $\phi' \circ i = \phi$.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "i"] \arrow[dr, "\phi"'] & A \arrow[d, dashed, "\exists! \phi'"] \\
& B
\end{tikzcd}$$

2. Denote by $C \coloneqq \sum X_i Y_i \in U(\mathfrak{g})$ the quadratic Casimir element of $\mathfrak{g}$, where $\{X_i\}, \{Y_i\}$ are bases of $\mathfrak{g}$, dual to each other with respect to the Killing form.

Lemma 1.6.2. *1. The universal enveloping algebra exists, and is unique up to isomorphism.*

2. The *quadratic Casimir element is well defined, and it lies in the center of* $U(\mathfrak{g})$.

Proof. [Hum78, Chapter II, §6.2 & §17.2]. ■

Remark 1.6.3. Observe that any representation ρ of $\mathfrak{g}$ can be extended to a representation $\hat{\rho}$ (i.e. an algebra morphism $U(\mathfrak{g}) \to \mathrm{End}(V)$) of $U(\mathfrak{g})$ by the definition. Now, if ρ is an irreducible representation of $\mathfrak{g}$, then $\hat{\rho}(C)$ commutes with the matrix representation of $\mathfrak{g}$. By Schur's lemma for algebraically closed fields, it has to be a scalar. We will determine this scalar for the irreducible representation of highest weight λ.

First we prove a special (but important) case.

Proposition 1.6.4. *Let the extension of* ad *to* $U(\mathfrak{g})$ *also be denoted by* ad. *Then* $\mathrm{ad}\, C$ *is the identity.*

Proof. The quadratic Casimir element C is in the center of $U(\mathfrak{g})$, so in particular it commutes with every element of $\mathfrak{g}$. This means that multiplication by C is a $\mathfrak{g}$-equivariant homomorphism of V to itself. By Schur's lemma (since k is algebraically closed), the morphism has to be a scalar μ. But we also know that

$$\mu \dim \mathfrak{g} = \mathrm{Tr}(\mathrm{ad}\, C) = \mathrm{Tr}(\sum_i \mathrm{ad}\, X_i \,\mathrm{ad}\, Y_i) = \sum_i K(X_i, Y_i) = \dim \mathfrak{g}.$$

Since we work over characteristic 0, this is only possible if $\mu = 1$. ■

We now have a proposition that tells us the Killing form is essentially the only bilinear form on $\mathfrak{g}$.

Proposition 1.6.5. *Let* $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle'$ *be two* $\mathfrak{g}$*-invariant non degenerate bilinear symmetric forms on* $\mathfrak{g}$. *Then they are equal up to a scalar.*

Proof. The adjoint representation of $\mathfrak{g}$ is irreducible. As both $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are non degenerate, the maps

$$\begin{aligned} \phi \colon \mathfrak{g} &\to \mathfrak{g}^* \\ v &\mapsto \langle v, \cdot \rangle \\ \phi' \colon \mathfrak{g} &\to \mathfrak{g}^* \\ w &\mapsto \langle w, \cdot \rangle' \end{aligned}$$

are $\mathfrak{g}$-equivariant isomorphisms. Denote $\psi = \phi^{-1} \circ \phi'$. Then ψ is a $\mathfrak{g}$-equivariant isomorphism of an irreducible representation. As k is algebraically closed, this means ψ is a scalar multiplication $\lambda \in k$ by Schur's Lemma. So it turns out that $\lambda \phi = \phi'$, or $\lambda \langle v, \cdot \rangle = \langle v, \cdot \rangle'$ for all $v \in \mathfrak{g}$. Equivalent but perhaps clearer, $\lambda \langle v, w \rangle = \langle v, w \rangle'$ for all $v, w \in \mathfrak{g}$. ■

We will make use of the following theorem from [Bou05].

Theorem 1.6.6 ([Bou05], VIII.§6.4). *Let* V *be an irreducible* $\mathfrak{g}$*-module* V *with highest weight* $\lambda \in \Lambda^+$. *Then* C *acts on* V *as scalar multiplication by* $\langle \lambda, \lambda + 2\delta \rangle$.

Proof. The quadratic Casimir element C lies in the center of $U(\mathfrak{g})$, so in particular it commutes with every element of $\mathfrak{g}$. This means that C acts on V as a $\mathfrak{g}$-equivariant endomorphism. By Schur's lemma (since k is algebraically closed), said morphism is a scalar multiplication.

Let $\mathfrak{h}$ be a Cartan subalgebra and $\mathcal{C}$ a Chevalley basis with respect to $\mathfrak{h}$. Now, let $H_1, \dots H_r$ be an orthonormal[3] basis of $\mathfrak{h}$. By Corollary 1.3.2 and Lemma 1.6.2, we have that

$$C = \sum_{\alpha \in \Phi} X_\alpha X_{-\alpha} \frac{1}{K(X_\alpha, X_{-\alpha})} + \sum_{i=1}^{r} H_i H_i.$$

We already know that C acts as a scalar on V. Let v be the maximal vector of V as in Theorem 1.4.8. We compute $C \cdot v$, and get

$$\begin{aligned}
C \cdot v &= \left(\sum_{\alpha \in \Phi} X_\alpha X_{-\alpha} \frac{1}{K(X_\alpha, X_{-\alpha})} + \sum_{i=1}^{r} H_i H_i \right) \cdot v \\
&= \left(\sum_{\alpha \in \Phi^+} X_\alpha X_{-\alpha} \frac{1}{K(X_\alpha, X_{-\alpha})} + \sum_{i=1}^{r} H_i H_i \right) \cdot v \\
&= \left(\sum_{\alpha \in \Phi^+} X_{-\alpha} X_\alpha \frac{1}{K(X_\alpha, X_{-\alpha})} + [X_\alpha, X_{-\alpha}] \frac{1}{K(X_\alpha, X_{-\alpha})} + \sum_{i=1}^{r} H_i H_i \right) \cdot v \\
&= \left(\sum_{\alpha \in \Phi^+} [X_\alpha, X_{-\alpha}] \frac{1}{K(X_\alpha, X_{-\alpha})} + \sum_{i=1}^{r} H_i H_i \right) \cdot v \\
&= \left(\sum_{\alpha \in \Phi^+} \lambda([X_\alpha, X_{-\alpha}]) \frac{1}{K(X_\alpha, X_{-\alpha})} + \sum_{i=1}^{r} \lambda(H_i)^2 \right) v.
\end{aligned}$$

To further simplify this, we recall Lemma 1.2.5 and Definition 1.2.6. Then it is readily seen that $\sum_i \lambda(H_i)^2 = \langle \lambda, \lambda \rangle$. The other term becomes

$$\begin{aligned}
&\sum_{\alpha \in \Phi^+} \lambda([X_\alpha, X_{-\alpha}]) \frac{1}{K(X_\alpha, X_{-\alpha})} \\
&= \sum_{\alpha \in \Phi^+} K(H'_\lambda, [X_\alpha, X_{-\alpha}]) \frac{1}{K(X_\alpha, X_{-\alpha})} \\
&= \sum_{\alpha \in \Phi^+} K([H'_\lambda, X_\alpha], X_{-\alpha}) \frac{1}{K(X_\alpha, X_{-\alpha})} \\
&= \sum_{\alpha \in \Phi^+} K(\alpha(H'_\lambda) X_\alpha, X_{-\alpha}) \frac{1}{K(X_\alpha, X_{-\alpha})} \\
&= \sum_{\alpha \in \Phi^+} \alpha(H'_\lambda) \\
&= \sum_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle \\
&= \langle \lambda, 2\delta \rangle.
\end{aligned}$$

In the end we have that $C \cdot v = \langle \lambda, \lambda + 2\delta \rangle v$. ■

Assembling this information, we can prove something useful. As mentioned in [CG21], this holds more generally, but for convenience we prove it over a field of characteristic zero.

[3] with respect to the Killing form of $\mathfrak{g}$

Proposition 1.6.7. *Let $\mathfrak{g}$ be a simple Lie algebra over k. Let $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ be an irreducible $\mathfrak{g}$-representation with highest weight λ. Denote by δ half the sum of the positive roots. Then:*

1. $\sum \pi(X_i)\pi(Y_i) = \langle \lambda, \lambda + 2\delta \rangle \cdot \mathrm{id}_V$
2. *For all $X, Y \in \mathfrak{g}$ we have* $\mathrm{Tr}(\pi(X)\pi(Y)) = \frac{\langle \lambda, \lambda+2\delta \rangle \cdot \dim V}{\dim \mathfrak{g}} K(X, Y)$.

Proof. We have already proven 1.

The map $(X, Y) \mapsto \mathrm{Tr}(\pi(X)\pi(Y))$ is a symmetric bilinear map. It remains to prove that it is non degenerate and $\mathfrak{g}$-equivariant.

The map is $\mathfrak{g}$-equivariant, since

$$\begin{aligned} \mathrm{Tr}(\pi([Z, X])\pi(Y)) &= \mathrm{Tr}((\pi(Z)\pi(X) - \pi(X)\pi(Z))\pi(Y)) \\ &= \mathrm{Tr}(\pi(Y)\pi(Z)\pi(X) - \pi(Z)\pi(Y)\pi(X)\ ) \\ &= -\mathrm{Tr}(\pi(X)\pi([Z, Y])) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$.

This also means the map is non degenerate. If S is the radical of the bilinear form, then by the $\mathfrak{g}$-equivariance, S would be an ideal of $\mathfrak{g}$. As $\mathfrak{g}$ is simple, it would either be 0 or $\mathfrak{g}$. It cannot be $\mathfrak{g}$, since $\chi(C) = \sum \mathrm{Tr}(\pi(X_i)\pi(Y_i)) = \langle \lambda, \lambda + 2\delta \rangle \cdot \dim V \neq 0$. So the radical S has to be 0.

Then 2 follows immediately from Proposition 1.6.4, Proposition 1.6.5 and Theorem 1.6.6. ■

Remark 1.6.8. Note that this is a different way of proving $\langle \tilde{\alpha}, \tilde{\alpha} + 2\delta \rangle = 1$, which we had already proven in Section 1.3.

1.7 Algebraic groups

We will try to avoid mentioning algebraic groups where possible, for the sake of simplicity. In essence they are matrix groups that satisfy certain polynomial equations. An abstract definition is as follows.

Definition 1.7.1. An (affine) algebraic group G over a field k is an affine group scheme over k. That is, G is an affine scheme over k (i.e., equipped with a morphism $\pi\colon G \to \mathrm{Spec}\, k$) with morphisms $m\colon G \times G \to G$ (multiplication), $\iota\colon G \to G$ (inverse), and $e\colon \mathrm{Spec}\, k \to G$ (identity) satisfying

$$\begin{aligned} m \circ (m \times \mathrm{id}_G) &= m \circ (\mathrm{id}_G \times m) \quad \text{(associativity)}, \\ m \circ (\iota \times \mathrm{id}_G) &= e \circ \pi = m \circ (\mathrm{id}_G \times \iota) \quad \text{(invertibility)}, \\ m \circ (e \times \mathrm{id}_G) &= \mu = m \circ (\mathrm{id}_G \times e) \quad \text{(identity element)}. \end{aligned}$$

Here $\mu\colon \mathrm{Spec}\, k \times G \tilde{\to} G$ denotes the natural isomorphism induced by $k \otimes_k k[G] \tilde{\to} k[G]$.

The unfamiliar reader can think of these as abstract groups, that are defined over a field, and can be base changed to other fields when necessary.

In the next chapter, we will mostly work with the corresponding Lie algebra. This is justified because we work over characteristic 0, and the following theorem.

Theorem 1.7.2. *Let G be a semisimple algebraic group over a field k of characteristic zero. Then the associated Lie algebra $\mathfrak{g}$ is semisimple, and the natural functor $\mathrm{Rep}(G) \to \mathrm{Rep}(\mathfrak{g})$ is fully faithful; it is essentially surjective if G is simply connected.*

Proof. [Mil17, Theorem 22.53]. ∎

Now, we do not assume G is simply connected, so the functor is not essentially surjective. However, all representations constructed start from the adjoint representation, which is in the essential image of the natural functor. In essence, this is why we can "pretend" to work with Lie algebras instead of algebraic groups.

In fact, algebraic groups are "up to isogeny" determined by the root systems associated to their Lie algebras, and are often also denoted by their associated root system. In the last four chapters we mention the group G_2, and then we mean the adjoint group with associated root system G_2.

2 The representation $A(\mathfrak{g})$

In this chapter, we construct a representation $A(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$. In characteristic zero, $\mathfrak{g}$ always becomes isomorphic to the Lie Algebra of type $A_l, B_l, C_l, D_l, G_2, F_4, E_6, E_7$ or E_8 for $l \in \mathbb{N}$ after extending the scalars to an algebraic closure of the field k. In positive characteristic however, this is not the case, see e.g. [PS08]. However, it is known from the theory of algebraic groups that this does hold for Lie algebras arising from reductive groups.

We will follow the approach of [CG21], though in [DMVC21] a different construction is given for the simply laced case.

Unless otherwise specified, we will assume $\operatorname{char} k = 0$. As mentioned before, [CG21] does this in more generality, but we will only need the characteristic zero case.

2.1 Constructing the underlying vector space

To construct $A(\mathfrak{g})$, we will start from the symmetric square $\mathrm{S}^2\,\mathfrak{g}$ of the Lie algebra. This carries a natural representation structure (see Example 1.4.4). However, the representation $\mathrm{S}^2\,\mathfrak{g}$ is too large, asthe resulting algebra will not be simple. To resolve this issue, Chayet and Garibaldi introduced an operator $S\colon \mathrm{S}^2\,\mathfrak{g} \to \operatorname{End}(\mathfrak{g})$. One could think of this map as a sort of *projection operator*.

Definition 2.1.1 ($A(\mathfrak{g})$ as a vector space).

- We define a linear map $S\colon \mathrm{S}^2\,\mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ by

$$S(XY) := h^\vee \operatorname{ad} X \bullet \operatorname{ad} Y + \frac{1}{2}(XK(Y, \_) + YK(X, \_)),$$

  where $h^\vee$ is the dual Coxeter number of the associated root system. This is well defined, since it is symmetric in X and Y.

- We define $A(\mathfrak{g}) := \operatorname{Im}(S)$.

Lemma 2.1.2. *The map S is $\mathfrak{g}$-equivariant.*

Proof. We prove that the two terms in the definition of S are equivariant. Recall that $L \cdot (XY) = [L, X]Y + X[L, Y]$. We compute $S(L \cdot (XY))$.

For the first term we get (ignoring constant coefficients):

$$\begin{aligned}
&(\operatorname{ad}[L, X] \circ \operatorname{ad} Y + \operatorname{ad} Y \circ \operatorname{ad}[L, X]) + (\operatorname{ad} X \circ \operatorname{ad}[L, Y] + \operatorname{ad}[L, Y] \circ \operatorname{ad} X) \\
&= (\operatorname{ad} L \circ \operatorname{ad} X - \operatorname{ad} X \circ \operatorname{ad} L) \circ \operatorname{ad} Y + \operatorname{ad} Y \circ (\operatorname{ad} L \circ \operatorname{ad} X - \operatorname{ad} X \circ \operatorname{ad} L) \\
&\quad + \operatorname{ad} X \circ (\operatorname{ad} L \circ \operatorname{ad} Y - \operatorname{ad} Y \circ \operatorname{ad} L) + (\operatorname{ad} L \circ \operatorname{ad} Y - \operatorname{ad} Y \circ \operatorname{ad} L) \circ \operatorname{ad} X \\
&= \operatorname{ad} L \circ (\operatorname{ad} X \circ \operatorname{ad} Y + \operatorname{ad} Y \circ \operatorname{ad} X) - (\operatorname{ad} X \circ \operatorname{ad} Y + \operatorname{ad} Y \circ \operatorname{ad} X) \circ \operatorname{ad} L \\
&= [\operatorname{ad} L, (\operatorname{ad} X \circ \operatorname{ad} Y + \operatorname{ad} Y \circ \operatorname{ad} X)]
\end{aligned}$$

by multiple applications of the Jacobi identity in the first equality.

The second term becomes (again ignoring constant coefficients):

$$\begin{aligned}&[L,X]K(Y,\_)+YK([L,X],\_)+[L,Y]K(X,\_)+XK([L,Y],\_)\\&=\operatorname{ad}L\circ(XK(Y,\_)+YK(X,\_))-(XK(Y,\_)+YK(X,\_))\circ\operatorname{ad}L\\&=[\operatorname{ad}L,(XK(Y,\_)+YK(X,\_))]\end{aligned}$$

by the $\mathfrak{g}$-equivariance of the Killing form. ■

Lemma 2.1.3. *Suppose $\mathfrak{g}$ is split with Chevalley basis $\mathcal{C}$. Let α and β be roots of $\mathfrak{g}$ such that $\alpha+\beta$ is not a root.*

1. *If $\langle\alpha,\beta\rangle=0$, then $S(X_\alpha X_\beta)\neq 0$.*
2. *Suppose $\langle\alpha,\beta\rangle>0$. Then $S(X_\alpha X_\beta)\neq 0$ in $A(\mathfrak{g})$ if and only if there are two root lengths and α and β are both short.*

Proof. Note that $\langle\alpha,-\alpha\rangle<0$. So we may assume that $\beta\neq-\alpha$. If $\langle\alpha,\beta\rangle=0$ holds, we know by Theorem 1.2.11 that $[X_\alpha,X_\beta]=0$. By applying the Jacobi identity, this means that $\operatorname{ad}X_\alpha\circ\operatorname{ad}X_\beta=\operatorname{ad}X_\beta\circ\operatorname{ad}X_\alpha$. Equivalently, $\operatorname{ad}X_\beta\bullet\operatorname{ad}X_\alpha=\operatorname{ad}X_\beta\circ\operatorname{ad}X_\alpha$ holds.

We compute the image of $X_{-\alpha}$ under $S(X_\alpha X_\beta)$ as (using Proposition 1.3.1)

$$\begin{aligned}S\left(X_\alpha X_\beta\right)\left(X_{-\alpha}\right)&=\left(\operatorname{ad}X_\beta\circ\operatorname{ad}X_\alpha+\frac{1}{2}\left(X_\alpha K\left(X_\beta,\_\right)+X_\beta K\left(X_\alpha,\_\right)\right)\right)\left(X_{-\alpha}\right)\\&=h^\vee[X_\beta,H_\alpha]+\frac{1}{2}\left(X_\alpha K\left(X_\beta,X_{-\alpha}\right)+X_\beta K\left(X_\alpha,X_{-\alpha}\right)\right)\\&=\left(\nu_\alpha-\alpha\left(\beta\right)\right)h^\vee X_\beta+\frac{1}{2}K\left(X_\beta,X_{-\alpha}\right)X_\alpha.\end{aligned}\tag{2.1}$$

If $\langle\alpha,\beta\rangle=0$, 2.1 becomes $\nu_\alpha h^\vee X_\beta$, which is non-zero. So in this case $S(X_\alpha X_\beta)$ is not the zero map. From now on we consider the case $\langle\alpha,\beta\rangle>0$.

Suppose $\alpha=\beta$. Then 2.1 becomes $(\nu_\alpha-\alpha(\alpha))h^\vee X_\alpha+\frac{1}{2}K(X_\alpha,X_{-\alpha})X_\alpha=(2\nu_\alpha-2)X_\alpha$. This is only zero if $\nu_\alpha=1$, i.e. if α is long.

If $\langle\alpha,\beta\rangle>0$, and α,β are different non-long[1] roots, then necessarily $\alpha(\beta)=1$ by Lemma 1.1.15. In this case, the coefficient of X_β is non-zero in 2.1.

Now suppose α is a long root, and $\alpha\neq\beta$. We will prove that $S(X_\alpha X_\beta)$ is the zero map by proving that it is the zero map on the Chevalley basis. Since the Weyl group $\mathcal{W}$ (See Definition 1.2.14) acts transitively on the long roots, and the fact that $S(w(XY))=wS(XY)w^{-1}$ for all $w\in\mathcal{W}$, we may assume α to be the highest root of the root system.

If $\beta\neq\alpha$, then $\alpha(\beta)=1$ by Lemma 1.1.15, so 2.1 is zero. By reversing the roles of α and β, we get

$$S(X_\alpha X_\beta)(X_{-\beta})=(\nu_\beta-\beta(\alpha))h^\vee X_\alpha.$$

Since α is the highest root, we know that $\beta(\alpha)=\nu_\beta$, thus $S(X_\alpha X_\beta)(X_{-\beta})=0$.

[1] this implies that there are 2 root lengths, and the roots we consider are short

For $H \in \mathfrak{h}$, we can compute that

$$\begin{aligned} S(X_\alpha X_\beta)(H) &= \left(h^\vee \operatorname{ad} X_\beta \circ \operatorname{ad} X_\alpha + \frac{1}{2}\left(X_\alpha K(X_\beta, \_) + X_\beta K(X_\alpha, \_)\right)\right)(H) \\ &= C_H[X_\beta, X_\alpha] \\ &= 0, \end{aligned}$$

with C_H a constant.

We now check that $S(X_\alpha X_\beta)(X_{-\gamma}) = 0$ for $\gamma \neq \alpha, \beta$:

$$\begin{aligned} S(X_\alpha X_\beta)(X_{-\gamma}) &= \left(h^\vee \operatorname{ad} X_\beta \circ \operatorname{ad} X_\alpha + \frac{1}{2}\left(X_\alpha K(X_\beta, \_) + X_\beta K(X_\alpha, \_)\right)\right)(X_{-\gamma}) \\ &= h^\vee [X_\beta, [X_\alpha, X_{-\gamma}]] + 0. \end{aligned}$$

This can only be nonzero if $\alpha - \gamma$ is a root in Φ, and $\alpha + \beta - \gamma$ is as well. As α is the highest root, this means we can assume γ is positive, and thus $\alpha(\gamma) \geq 0$ by Lemma 1.1.15. If we have $\alpha(\gamma) = 0$, then $\alpha - \gamma - (\gamma(\alpha - \gamma))\gamma = \alpha + \gamma$ has to be a root, a contradiction.

If $\alpha(\gamma) > 0$, then we have $\alpha(\gamma) = \alpha(\beta) = 1$ since $\alpha \neq \gamma$. Then $\alpha(\beta - \gamma) = 0$. So if $\alpha + \beta - \gamma$ is a root in Φ, then so is $\alpha + \beta - \gamma - ((\gamma - \beta)(\alpha + \beta - \gamma))(\gamma - \beta) = \alpha - \beta + \gamma$. Since either $\beta - \gamma$ or $\gamma - \beta$ is a positive root, this is absurd. We have finally proved that $S(X_\alpha, X_\beta)$ is the zero map. ■

This lemma is important for the following corollary, and hints towards Proposition 2.6.1.

Corollary 2.1.4. *$2\tilde{\alpha}$ is not a weight of $A(\mathfrak{g})$, with $\tilde{\alpha}$ the highest root of the associated root system.*

Proof. Denote by $\mathcal{C}$ the Chevalley basis of $\mathfrak{g}$. Then $\{XY \mid X, Y \in \mathcal{C}\}$ is a basis of weight vectors of the representation $\mathrm{S}^2\,\mathfrak{g}$. This shows us that the $2\tilde{\alpha}$ weight space of $\mathrm{S}^2\,\mathfrak{g}$ is generated by $X_{\tilde{\alpha}} X_{\tilde{\alpha}}$.

But S is $\mathfrak{g}$-equivariant, and $S(X_{\tilde{\alpha}} X_{\tilde{\alpha}}) = 0$ by the previous Lemma. This proves the statement. ■

2.2 A product on $A(\mathfrak{g})$

To construct an algebra, we of course need to describe a product on the vector space from the previous section.

Definition 2.2.1 (The product $\diamond$). We define a product $\diamond$: $A(\mathfrak{g}) \times A(\mathfrak{g}) \to A(\mathfrak{g})$ by

$$\begin{aligned} S(AB) \diamond S(CD) := \frac{h^\vee}{2}&\big(S(A, (\operatorname{ad} C \bullet \operatorname{ad} D)B) + S((\operatorname{ad} C \bullet \operatorname{ad} D)A, B) \\ &+ S(C, (\operatorname{ad} A \bullet \operatorname{ad} B)D) + S((\operatorname{ad} A \bullet \operatorname{ad} B)C, D) \\ &+ S([A, C][B, D]) + S([A, D][B, C])\big) \\ + \frac{1}{4}&\big(K(A, C)S(BD) + K(A, D)S(BC) \\ &+ K(B, C)S(AD) + K(B, D)S(AC)\big) \quad \forall A, B, C, D \in \mathfrak{g}, \end{aligned}$$

and linearly extending this product in both terms.

Notice that this product is not necessarily well-defined. We prove this now.

Lemma 2.2.2. *The product operation* $\diamond$ *is well-defined and* $\mathfrak{g}$*-equivariant.*

Proof. 1. We need to make sure that every product of two elements in $A(\mathfrak{g})$ has precisely one image. It suffices to prove that if $v \in \ker(S)$, the equality

$$S(w) \diamond S(v) = 0 \tag{2.2}$$

holds for any $w \in \mathrm{S}^2\,\mathfrak{g}$.

We can reduce this question to the case where k is algebraically closed. Indeed, denote by $\overline{k}$ its algebraic closure, and when we index a structure defined over k by $\overline{k}$ we mean by that its lift to $\overline{k}$. If we prove that $S_{\overline{k}}(w) \diamond S_{\overline{k}}(v) = 0$ for all $v \in \ker(S_{\overline{k}})$ and all $w \in \mathrm{S}^2\,\mathfrak{g}_{\overline{k}}$, Then by restricting v and w to be in $\mathrm{S}^2\,\mathfrak{g}$ (viewed as a subset of $\mathrm{S}^2\,\mathfrak{g}_{\overline{k}}$), we obtain what we originally had to prove.

It is also sufficient to check the condition when $w = Y^2$ for any $Y \in \mathfrak{g}$. Then the condition holds for $w = XY$ by checking the condition for $(X+Y)^2$. But if it holds for any XY, it holds for all $w \in \mathrm{S}^2\,\mathfrak{g}$ by linearity.

We can make one more reduction before making the computations. As k is algebraically closed, there is an element $\mathbf{i} \in k$ such that $\mathbf{i}^2 = -1$. Suppose we have an element of $\mathrm{S}^2\,\mathfrak{g}$ of the form aXY, with $X, Y \in \mathfrak{g}$. Then we have $aXY = \left(\frac{a}{2}X + Y\right)^2 + \left(\frac{a\mathbf{i}}{2}X\right)^2 + (\mathbf{i}Y)^2$. In other words, we can write any $v \in \mathrm{S}^2\,\mathfrak{g}$ as $v = \sum_i X_i^2$ for certain $X_i \in \mathfrak{g}$.

Armed with this knowledge, we compute 2.2 with $v = \sum_i X_i^2$ and $w = Y^2$ for $Y, X_i \in \mathfrak{g}$ such that $v \in \ker S$:

$$\begin{aligned} S(w) \diamond S(v) &= \sum S(Y^2) \diamond S(X_i^2) \\ &= h^\vee \sum S(((\operatorname{ad} Y)^2 X_i) X_i) + h^\vee \sum S(((\operatorname{ad} X_i)^2 Y) Y) \\ &\quad + h^\vee \sum S([Y, X_i][Y, X_i]) + \sum K(Y, X_i) S(X_i Y). \end{aligned} \tag{2.3}$$

Notice that $S(v) = \sum \big(h^\vee (\operatorname{ad} X_i)^2 + X_i K(X_i, \_)\big) = 0$. Using this, we get

$$\begin{aligned} S(w) \diamond S(v) &= h^\vee \sum S(((\operatorname{ad} Y)^2 X_i) X_i) + h^\vee \sum S([Y, X_i][Y, X_i]) \\ &\quad + \sum S((S(v)(Y))Y) \\ &= h^\vee \sum S(((\operatorname{ad} Y)^2 X_i) X_i) + h^\vee \sum S([Y, X_i][Y, X_i]) \\ &= h^\vee \sum S([Y, [Y, X_i]] X_i + [Y, X_i][Y, X_i]) \\ &= h^\vee \sum [\operatorname{ad} Y, S([Y, X_i] X_i)] \\ &= h^\vee \sum \left[\operatorname{ad} Y, \frac{1}{2} S([Y, X_i]\, X_i + X_i\, [Y, X_i])\right] \\ &= \frac{h^\vee}{2} \sum \left[\operatorname{ad} Y, \left[\operatorname{ad} Y, S(X_i^2)\right]\right] \\ &= \frac{h^\vee}{2} [\operatorname{ad} Y, [\operatorname{ad} Y, S(v)]] \\ &= 0. \end{aligned}$$

Here we used the $\mathfrak{g}$-equivariance of S exactly two times.

2. To check $\diamond$ is $\mathfrak{g}$-equivariant, it suffices to check that

$$(L \cdot a) \diamond a' + a \diamond (L \cdot a') = L \cdot (a \diamond a')$$

for $a = S(X^2), a' = S(Y^2)$, with $X, Y, L \in \mathfrak{g}$. The reasoning for this is exactly the same as in the previous paragraph. For the first term we get

$$\begin{aligned}(L \cdot a) \diamond a' &= S\left(2\left[L, X\right], X\right) \diamond S\left(Y^2\right) \\ &= h^\vee \left(S\left((\operatorname{ad} Y)^2\left[L, X\right], X\right) + S\left([L, X], (\operatorname{ad} Y)^2 X\right)\right) && (2.4)\\ &\quad + 2h^\vee S((\operatorname{ad}[L, X] \bullet \operatorname{ad} X)Y, Y) && (2.5)\\ &\quad + 2h^\vee S\left(\left[\left[L, X\right], Y\right], \left[X, Y\right]\right) && (2.6)\\ &\quad + K\left(\left[L, X\right], Y\right) S(X, Y) + K(X, Y) S\left(\left[L, X\right], Y\right). && (2.7)\end{aligned}$$

We will manipulate each of these terms separately. To keep things interesting, we will first calculate 2.5. Note first that

$$\begin{aligned}2 \operatorname{ad}[L, X] \bullet \operatorname{ad} X &= 2[\operatorname{ad} L, \operatorname{ad} X] \bullet \operatorname{ad} X \\ &= (\operatorname{ad} L \operatorname{ad} X - \operatorname{ad} X \operatorname{ad} L) \operatorname{ad} X + \operatorname{ad} X(\operatorname{ad} L \operatorname{ad} X - \operatorname{ad} X \operatorname{ad} L) \\ &= \operatorname{ad} L(\operatorname{ad} X)^2 - (\operatorname{ad} X)^2 \operatorname{ad} L.\end{aligned}$$

Then 2.5 becomes

$$\begin{aligned}2h^\vee S((\operatorname{ad}[L, X] \bullet \operatorname{ad} X)Y, Y) &= h^\vee \left(S\left(\left(\operatorname{ad} L(\operatorname{ad} X)^2\right) Y, Y\right) - S\left(\left((\operatorname{ad} X)^2 \operatorname{ad} L\right) Y, Y\right)\right) \\ &= h^\vee L \cdot S((\operatorname{ad} X)^2 Y, Y) \\ &\quad - h^\vee (S((\operatorname{ad} X)^2 Y, (\operatorname{ad} L) Y) + S((\operatorname{ad} X)^2 (\operatorname{ad} L) Y, Y)).\end{aligned}$$

Now we compute the other terms in order. The term 2.4 becomes

$$\begin{aligned}&h^\vee (S((\operatorname{ad} Y)^2 [L, X], X) + S([L, X], (\operatorname{ad} Y)^2 X)) \\ &= h^\vee (S((\operatorname{ad} Y)^2 [L, X], X) + L \cdot S(X, (\operatorname{ad} Y)^2 X) - S((\operatorname{ad} L(\operatorname{ad} Y)^2) X)) \\ &= h^\vee L \cdot S(X, (\operatorname{ad} Y)^2 X) \\ &\quad - 2h^\vee S((\operatorname{ad}[L, Y] \bullet \operatorname{ad} Y) X, X).\end{aligned}$$

In the last equality, we conveniently reuse the computation we made for 2.5.

For the third term 2.6 we use the Jacobi identity:

$$\begin{aligned}2h^\vee S([[L, X], Y], [X, Y]) &= 2h^\vee (S([X, [Y, L]], [X, Y]) + S([L, [X, Y]], [X, Y])) \\ &= h^\vee L \cdot S([X, Y], [X, Y]) \\ &\quad - 2h^\vee S([[L, Y], X], [Y, X]).\end{aligned}$$

The last term 2.7 simplifies to

$$\begin{aligned}&K([L, X], Y) S(X, Y) + K(X, Y) S([L, X], Y) \\ &= L \cdot (K(X, Y) S(X, Y)) \\ &\quad - (K([L, Y], X) S(Y, X) + K(X, Y) S([L, Y], X)).\end{aligned}$$

Comparing these terms with the original terms we had and 2.3, we obtain that $\diamond$ is $\mathfrak{g}$-equivariant. ■

Remark 2.2.3. To prove this product is $\mathfrak{g}$-equivariant, it is actually a lot easier to prove that it is G-equivariant, and then use the correspondence between G and $\mathfrak{g}$ (See Theorem 1.7.2). As mentioned before, we choose to construct everything in terms of the Lie algebra in this chapter, for simplicity.

The algebra constructed turns out to be unital.

Lemma 2.2.4. *The identity transformation $e = \mathrm{id}_{\mathfrak{g}}$ of $\mathfrak{g}$ is the multiplicative identity in $A(\mathfrak{g})$.*

Proof. First of all, we need to prove that e is in $A(\mathfrak{g})$. This holds by Proposition 1.6.4. Indeed, let $\{X_i\}$ be a basis of $\mathfrak{g}$ with dual basis $\{Y_i\}$, with respect to the Killing form. Then $S(\sum_i X_iY_i) = h^\vee e + \sum_i \frac{1}{2}(X_iK(Y_i, \_) + Y_iK(X_i, \_))$, by Proposition 1.6.4. The second term acts as the identity on the basis $\{X_i\}$, so it is equal to the identity.

As in the previous lemma, we may assume k to be algebraically closed. If this is the case, $\mathfrak{g}$ has an orthonormal basis $\{X_i\}$ with respect to the nondegenerate bilinear form K. As in the previous lemma, we only have to check

$$e \diamond S(Y^2) = S(Y^2). \tag{2.8}$$

This is proved in a very similar way to the previous proposition, and we reuse some of the calculations.

$$\begin{aligned}
(h^\vee + 1)e \diamond S(Y^2) &= \sum_i S(X_i^2) \diamond S(Y^2) \\
&= \frac{h^\vee}{2} \sum_i [\operatorname{ad} Y, [\operatorname{ad} Y, S(X_i^2)]] \\
&\quad + h^\vee S((\operatorname{ad} X_i)^2 Y, Y) + K(X_i, Y)S(X_iY) \\
&= \frac{h^\vee}{2} [\operatorname{ad} Y, [\operatorname{ad} Y, S(\sum_i X_i^2)]] \\
&\quad + h^\vee S(\sum_i (\operatorname{ad} X_i)^2 Y, Y) + \sum_i K(X_i, Y)S(X_iY) \\
&= 0 + h^\vee S(Y^2) + S(Y^2).
\end{aligned}$$

Dividing both sides by $h^\vee + 1$ shows e is the identity. ∎

2.3 $A(\mathfrak{g})$ as an algebra adjoined by a unit

To be able to prove structural properties of this algebra, we will introduce a counit and a bilinear form on the algebra.

Definition 2.3.1 (The counit). Define the algebra morphism $\varepsilon \colon A(\mathfrak{g}) \to k$ by

$$\varepsilon(a) := \frac{\operatorname{Tr}(a)}{\dim \mathfrak{g}},$$

where we regard a as an endomorphism of $\mathfrak{g}$. Then ε is a counit for $A(\mathfrak{g})$, since $\varepsilon(e) = 1$.

Using this counit we can decompose the algebra $A(\mathfrak{g}) = ke \oplus V$, where V is the kernel of ε. The counit ε is $\mathfrak{g}$-equivariant in the sense that $\varepsilon(X \cdot a) = 0$ for all $X \in \mathfrak{g}$ and $a \in A(\mathfrak{g})$, so V is a $\mathfrak{g}$-invariant subspace. In this way, we can construct $A(\mathfrak{g})$ as the representation V, adjoined by a unit (see appendix A of [CG21]).

It turns out this multiplication is far from being associative, it is not even *power associative*.

Proposition 2.3.2. *If $\mathfrak{g}$ is not of type A_1 nor A_2, then:*

1. *The multiplication on V is not zero.*

2. *Neither V nor $\mathcal{U}(V, cf)$ are power-associative for any $c \in k$.*

Proof. See [CG21, Proposition 5.3]. ∎

2.4 Jordan subalgebras of $A(\mathfrak{g})$

One notes from Definition 2.2.1 that any subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ gives rise to a subalgebra of $A(\mathfrak{g})$. In this section we look at some properties of these structures if $\mathfrak{l}$ is abelian, such as when $\mathfrak{l}$ is a Cartan subalgebra.

Notation 2.4.1. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{l} \leq \mathfrak{g}$ a subalgebra. We denote by

$$A(\mathfrak{g}, \mathfrak{l}) := S(\mathrm{S}^2\, \mathfrak{l})$$

the image of $\mathrm{S}^2\, \mathfrak{l}$, viewed as a subalgebra of $\mathrm{S}^2\, \mathfrak{g}$.

Proposition 2.4.2. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{l} \leq \mathfrak{g}$ an abelian subalgebra such that the Killing form is nondegenerate on $\mathfrak{l}$. Then the restriction*

$$|_{\mathfrak{l}} \colon A(\mathfrak{g}, \mathfrak{l}) \to \mathcal{H}(\mathfrak{l})$$

is an algebra isomorphism, where by $\mathcal{H}(\mathfrak{l})$ we mean the Jordan algebra of symmetric operators with respect to the Killing form of $\mathfrak{g}$.

Proof. Restriction is clearly a linear map.

For $A, B \in \mathfrak{l}$, we have that

$$S(AB)|_{\mathfrak{l}} = \frac{1}{2}(AK_{\mathfrak{g}}(B, \_) + BK_{\mathfrak{g}}(A, \_))$$

since $\mathfrak{l}$ is abelian. As the Killing form is nondegenerate on $\mathfrak{l}$, the restriction is injective. By definition of $\mathcal{H}(\mathfrak{l})$, the map is also surjective.

It remains to show that the restriction preserves the product structure. It suffices to check this for elements of the form $S(AB), S(CD)$.

Because the subalgebra $\mathfrak{l}$ is abelian, the first terms in the product from Definition 2.2.1 vanish. So for $A, B, C, D \in \mathfrak{l}$ we have that

$$S(AB) \diamond S(CD) = \frac{1}{4}(K(A, C)S(BD) + K(A, D)S(BC) + K(B, C)S(AD) + K(B, D)S(AC))$$

and

$$\begin{aligned}
S(AB)|_{\mathfrak{l}} \bullet S(CD)|_{\mathfrak{l}} &= \frac{1}{2}(S(AB)|_{\mathfrak{l}} S(CD)|_{\mathfrak{l}} + S(CD)|_{\mathfrak{l}}(AB)|_{\mathfrak{l}}) \\
&= \frac{1}{8}K(A,C)(BK(D,_) + DK(B,_)) \\
&\quad + \frac{1}{8}K(B,C)(AK(D,_) + DK(A,_)) \\
&\quad + \frac{1}{8}K(A,D)(BK(C,_) + CK(B,_)) \\
&\quad + \frac{1}{8}K(B,D)(AK(C,_) + CK(A,_)) \\
&= \frac{1}{4}(K(A,C)S(BD)|_{\mathfrak{l}} + K(A,D)S(BC)|_{\mathfrak{l}} \\
&\quad + K(B,C)S(AD)|_{\mathfrak{l}} + K(B,D)S(AC)|_{\mathfrak{l}}).
\end{aligned}$$

Comparing these two computations, we see the product is preserved. ∎

Remark 2.4.3. The structure of these Jordan algebras has been studied in [DMR17, Section 3].

2.5 Associativity of the bilinear form τ

Using the counit ε we can also define a bilinear form on the algebra.

Definition 2.5.1 (The bilinear form τ). Define the bilinear form $\tau\colon A(\mathfrak{g}) \times A(\mathfrak{g}) \to k$ by

$$\tau(a, a') := \varepsilon(a \diamond a').$$

Note that τ is $\mathfrak{g}$-equivariant, i.e.

$$\tau(L \cdot a, a') + \tau(a, L \cdot a') = 0 \quad \text{for all } L \in \mathfrak{g} \text{ and } a, a' \in A(\mathfrak{g}).$$

In this section, we prove that τ is compatible in some way with the product $\diamond$ defined on $A(\mathfrak{g})$. This is not trivial, as Proposition 2.3.2 shows.

Lemma 2.5.2 (Associativity of τ). *The bilinear form τ on $A(\mathfrak{g})$ satisfies*

$$\tau(a \diamond a', a'') = \tau(a, a' \diamond a'') \quad \textit{for all } a, a', a'' \in A(\mathfrak{g}).$$

Proof. Again, if this holds after base changing to an algebraic closure, this also holds over the base field k. As mentioned before, any element in $\mathrm{S}^2\,\mathfrak{g}$ over an algebraic closure can be written as the sum of squares, so we verify the statement for $a = S(X^2), a' = S(Y^2), a'' = S(Z^2)$ for $X, Y, Z \in \mathfrak{g}$. If this holds, then by linearity the statement holds over the whole of $A(\mathfrak{g})$.

We will calculate $\tau(a \diamond a', a'')$ in a couple steps.

1. First we need to compute $\varepsilon(S(XY))$ for $X, Y \in \mathfrak{g}$.

   To do this, we need to calculate the trace of an endomorphism of the form $\phi = XK(Y,\_)$. Extending X to a basis of $\mathfrak{g}$, one can immediately see that with respect to this basis, ϕ has trace $K(Y, X)$. We then have

$$\begin{aligned}
\varepsilon(S(XY)) &= \frac{1}{\dim \mathfrak{g}} \operatorname{Tr}\left(\frac{h^\vee}{2}(\operatorname{ad} X \circ \operatorname{ad} Y + \operatorname{ad} Y \circ \operatorname{ad} X) + \frac{1}{2}(XK(Y,_) + YK(X,_))\right) \\
&= \frac{h^\vee + 1}{\dim \mathfrak{g}} K(X, Y).
\end{aligned}$$

2. With this knowledge, we can calculate $\tau(S(XY), S(Z^2))$ for all $X, Y, Z \in \mathfrak{g}$. We get

$$\begin{aligned}
\tau(S(XY), S(Z^2)) &= \varepsilon(S(XY) \diamond S(Z^2)) \\
&= \varepsilon\Bigg(\frac{h^\vee}{2}(S((\operatorname{ad} Z)^2 X, Y) + S(X, (\operatorname{ad} Z)^2 Y)) \\
&\quad + h^\vee S((\operatorname{ad} X \bullet \operatorname{ad} Y)Z, Z) + h^\vee S([X, Z], [Y, Z]) \\
&\quad + \frac{1}{2}K(X, Z)S(Y, Z) + \frac{1}{2}K(Y, Z)S(X, Z)\Bigg) \\
&= \frac{h^\vee + 1}{\dim \mathfrak{g}}\Bigg(\frac{h^\vee}{2}(K((\operatorname{ad} Z)^2 X, Y) + K(X, (\operatorname{ad} Z)^2 Y)) \\
&\quad + h^\vee K((\operatorname{ad} X \bullet \operatorname{ad} Y)Z, Z) + h^\vee K([X, Z], [Y, Z]) \\
&\quad + \frac{1}{2}K(X, Z)K(Y, Z) + \frac{1}{2}K(Y, Z)K(X, Z)\Bigg).
\end{aligned}$$

We can simplify this using the $\mathfrak{g}$-equivariance of the Killing form. The middle two terms in the last formula become

$$\begin{aligned}
&h^\vee K((\operatorname{ad} X \bullet \operatorname{ad} Y)Z, Z) + h^\vee K([X, Z], [Y, Z]) \\
&= \frac{h^\vee}{2}(K((\operatorname{ad} X \operatorname{ad} Y)Z, Z) + K((\operatorname{ad} Y \operatorname{ad} X)Z, Z)) + h^\vee K((\operatorname{ad} X)Z, (\operatorname{ad} Y)Z) \\
&= -\frac{h^\vee}{2}\left(K\left((\operatorname{ad} X)Z, (\operatorname{ad} Y)Z\right) + K\left((\operatorname{ad} Y)Z, (\operatorname{ad} X)Z\right)\right) + h^\vee K((\operatorname{ad} X)Z, (\operatorname{ad} Y)Z) \\
&= 0.
\end{aligned}$$

In the end, we get that

$$\begin{aligned}
\tau(S(XY), S(Z^2)) = \frac{h^\vee + 1}{\dim \mathfrak{g}}&\left(h^\vee K((\operatorname{ad} Z)^2 X, Y)\right. \\
&\left. + K(X, Z)K(Y, Z)\ \right).
\end{aligned}$$

3. Lastly, we calculate $\tau(a \diamond a', a'')$ using step 2:

$$\begin{aligned}
\frac{\dim \mathfrak{g}}{h^\vee + 1}\tau\left(a \diamond a', a''\right) &= \frac{\dim \mathfrak{g}}{h^\vee + 1}\tau\Big(h^\vee S\Big((\operatorname{ad} Y)^2 X, X\Big) + h^\vee S\Big((\operatorname{ad} X)^2 Y, Y\Big) \\
&\quad + h^\vee S\left([Y, X], [Y, X]\right) + K\left(X, Y\right) S\left(XY\right), S\left(Z^2\right)\Big) \\
&= \left(h^\vee\right)^2 K\Big((\operatorname{ad} Z)^2 (\operatorname{ad} Y)^2 X, X\Big) + h^\vee K\Big((\operatorname{ad} Y)^2 X, Z\Big) K\left(X, Z\right) \\
&\quad + \left(h^\vee\right)^2 K\Big((\operatorname{ad} Z)^2 (\operatorname{ad} X)^2 Y, Y\Big) + h^\vee K\Big((\operatorname{ad} X)^2 Y, Z\Big) K\left(Y, Z\right) \\
&\quad + \left(h^\vee\right)^2 K\Big((\operatorname{ad} Z)^2 \left([Y, X]\right), [Y, X]\Big) + h^\vee K\left(Z, [Y, X]\right)^2 \\
&\quad + h^\vee K\left(X, Y\right) K\Big((\operatorname{ad} Z)^2 X, Y\Big) + K\left(X, Z\right) K\left(Y, Z\right) K\left(X, Y\right) \\
&= \left(h^\vee\right)^2 \Big(K\Big((\operatorname{ad} Y)^2 X, (\operatorname{ad} Z)^2 X\Big) + K\Big((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\Big) \\
&\qquad + K\Big((\operatorname{ad} Z)^2 \left([Y, X]\right), [Y, X]\Big)\Big) \\
&\quad + h^\vee \Big(K\Big((\operatorname{ad} Y)^2 X, Z\Big) K\left(X, Z\right) + K\Big((\operatorname{ad} X)^2 Y, Z\Big) K\left(Y, Z\right) \\
&\qquad + K\left(X, Y\right) K\Big((\operatorname{ad} Z)^2 X, Y\Big) + K\left(Z, [Y, X]\right)^2\Big) \\
&\quad + K\left(X, Z\right) K\left(Y, Z\right) K\left(X, Y\right).
\end{aligned}$$

The last terms we can simplify are the coefficient of $(h^\vee)^2$ in the formula:

$$
\begin{aligned}
&K\left((\operatorname{ad} Y)^2 X, (\operatorname{ad} Z)^2 X\right) + K\left((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\right) + K\left((\operatorname{ad} Z)^2([Y,X]), [Y,X]\right) \\
&= K\left((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\right) \\
&\quad + K\left(\left(\operatorname{ad} Y (\operatorname{ad} Z)^2\right) X, [X,Y]\right) + K\left((\operatorname{ad} Z)^2([X,Y]), [X,Y]\right) \\
&= K\left((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\right) \\
&\quad - K\left(\operatorname{ad}\left((\operatorname{ad} Z)^2 X\right) Y, [X,Y]\right) + K\left((\operatorname{ad} Z)^2 (\operatorname{ad} X) Y, [X,Y]\right) \\
&= K\left((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\right) \\
&\quad + K\left(\left(-[\operatorname{ad} Z, [\operatorname{ad} Z, \operatorname{ad} X]] + \left((\operatorname{ad} Z)^2 \operatorname{ad} X\right)\right) Y, [X,Y]\right) \\
&= K\left((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\right) \\
&\quad + K\left(\left(-(\operatorname{ad} Z)^2(\operatorname{ad} X) + 2(\operatorname{ad} Z \operatorname{ad} X \operatorname{ad} Z) - (\operatorname{ad} X)(\operatorname{ad} Z)^2 + (\operatorname{ad} Z)^2(\operatorname{ad} X)\right) Y, [X,Y]\right) \\
&= K\left((\operatorname{ad} X)^2 Y, (\operatorname{ad} Z)^2 Y\right) + 2K\left((\operatorname{ad} X \operatorname{ad} Z) Y, (\operatorname{ad} Z \operatorname{ad} X) Y\right) - K\left((\operatorname{ad} Z)^2 Y, (\operatorname{ad} X)^2 Y\right) \\
&= 2K\left((\operatorname{ad} X \operatorname{ad} Z) Y, (\operatorname{ad} Z \operatorname{ad} X) Y\right).
\end{aligned}
$$

To summarise, we get

$$
\begin{aligned}
\frac{\dim \mathfrak{g}}{h^\vee + 1}\tau\left(a \diamond a', a''\right) = {}& 2K\left((\operatorname{ad} X \operatorname{ad} Z) Y, (\operatorname{ad} Z \operatorname{ad} X) Y\right)\left(h^\vee\right)^2 \\
&+ \left(K\left((\operatorname{ad} Y)^2 X, Z\right) K(X,Z) + K\left((\operatorname{ad} X)^2 Y, Z\right) K(Y,Z)\right. \\
&\left. + K\left((\operatorname{ad} Z)^2 X, Y\right) K(X,Y) + K(Z,[Y,X])^2\right) h^\vee \\
&+ K(X,Z) K(Y,Z) K(X,Y).
\end{aligned} \tag{2.9}
$$

It is clear that τ is associative if this last formula is symmetric in X and Z. The last term is easily seen to be symmetric in X and Z. If we look at the coefficient belonging to $h^\vee$, The first three terms together are symmetric in X and Z. The last term is equal to itself when we swap X and Z:

$$K(Z,[Y,X])^2 = (-K([Y,Z],X))^2 = K(X,[Y,Z])^2.$$

The coefficient of $(h^\vee)^2$ is symmetric in X and Z as well. ■

Remark 2.5.3. Sometimes, algebras equipped with a bilinear form that associates with the algebra product are called *Frobenius algebras* (as in [DMVC21]) or *metrized algebras* (as in [CG21]).

2.6 Understanding the module structure on $A(\mathfrak{g})$

In this section, we will state some of the central structure theorems in the paper by Chayet & Garibaldi. As a first theorem, they analyzed the explicit action of $\mathfrak{g}$ on the constructed algebra. As e is the unit of $\diamond$, and $\diamond$ is $\mathfrak{g}$-equivariant, it suffices to analyze V as a $\mathfrak{g}$-module.

Proposition 2.6.1 ([CG21], Proposition 7.2). *Let $\mathfrak{g}$ be a Lie algebra associated to an absolutely simple algebraic group G,* $\operatorname{char} k = 0$ *or*

$$\operatorname{char} k \geq \frac{\binom{\dim(\mathfrak{g})+1}{2}}{\operatorname{rk}(\mathfrak{g})}.$$

As a representation of G, $A(\mathfrak{g})$ is a direct sum of pairwise non-isomorphic irreducible modules and $\mathcal{H}(\mathfrak{g}) = A(\mathfrak{g}) \oplus V(2\tilde{\alpha})$. Furthermore, if G and λ are as in [CG21, Table 1] (copied into table 2.1), then $A(\mathfrak{g}) = k \oplus L(\lambda)$.

type of G	A_2	G_2	F_4	E_6	E_7	E_8
dual Coxeter number $h^\vee$	3	4	9	12	18	30
Coxeter number h	3	6	12	12	18	30
Dominant weight λ	$\omega_1 + \omega_2$	$2\omega_1$	$2\omega_4$	$\omega_1 + \omega_6$	ω_6	ω_1
Dim. of irred. rep $L(\lambda)$	8	27	324	650	1539	3875

Table 2.1: The table from [CG21]. The fundamental dominant weights are labelled using Bourbaki labelling.

The previous proposition gives us some information about the size of the algebra.

Corollary 2.6.2. *Under the assumptions of the previous proposition, we have*

$$\dim A(\mathfrak{g}) = \binom{\dim(\mathfrak{g}) + 1}{2} - \dim(V(2\tilde{\alpha})).$$

Note that the dimension of highest weight modules can be calculated using Weyl's dimension formula.

Using Proposition 2.6.1, Chayet and Garibaldi proved the following:

Corollary 2.6.3 ([CG21], Proposition 8.1). *The bilinear form τ is nondegenerate on $A(\mathfrak{g})$.*

We will use the nondegenerateness of the bilinear form to narrow down the option for automorphisms of the algebra.

The algebra turns out to be simple as well.

Proposition 2.6.4 ([CG21], Corollary 8.6). *The algebra $A(\mathfrak{g})$ is simple.*

In the next chapters, we diverge from [CG21], and use the observations from their paper to prove some additional results in the case of G_2.

3 The octonion algebra and $A(\mathfrak{g}_2)$

In this chapter, we again work over fields of characteristic zero. This is only because of the representation theoretic arguments in Proposition 3.3.8. However, as mentioned before, these results should in principle also hold over more general fields.

The idea of this chapter is to provide a simpler description of the algebra $A(\mathfrak{g}_2)$, in hopes of gaining better insight into its structure. We will rely on the octonions to do the heavy lifting.

3.1 The mother

Chayet and Garibaldi noticed we can embed the representation constructed in their paper in the endomorphism ring of the "natural" representation, which relates the algebra to a much smaller representation of the corresponding algebraic group.

Proposition 3.1.1. *If G has type A_2, G_2, F_4, E_6 or E_7 and $\pi\colon G \to \mathrm{GL}(V)$ is the natural irreducible representation of dimension $3, 7, 26, 27$ or 56 respectively, then the formula*

$$\sigma(S(XY)) = 6h^\vee \pi(X) \bullet \pi(Y) - \frac{1}{2}K(X,Y) \tag{3.1}$$

defines an injective G-equivariant linear map

$$\sigma\colon A(\mathfrak{g}) \hookrightarrow \mathrm{End}(V).$$

Moreover, σ maps the identity $\mathrm{id}_{\mathfrak{g}}$ to the identity id_V.

Proof. See [CG21, Proposition 10.5]. The fact that σ maps the identity of $\mathfrak{g}$ to the identity of V, follows from the proof as well. More specifically, in these cases specifically we have

$$\langle \lambda, \lambda + 2\delta \rangle = \frac{h^\vee + 1}{h^\vee + 6}.$$

Then, for dual bases $\{X_i\}$ and $\{Y_i\}$ of $\mathfrak{g}$ we have

$$\begin{aligned}\sigma((\sum_i X_i Y_i)) &= \sum_i 6h^\vee \pi(X_i) \bullet \pi(Y_i) - \frac{1}{2}K(X_i, Y_i)\\ &= 6h^\vee \frac{h^\vee + 1}{h^\vee + 6} - \frac{\dim \mathfrak{g}}{2}.\end{aligned}$$

By [CdM96, p.431], we have for all the cases in the proposition the equality

$$\dim \mathfrak{g} = 2\frac{(5h^\vee - 6)(1 + h^\vee)}{6 + h^\vee}.$$

Thus we get

$$\begin{aligned}\sigma(S(\sum_i X_iY_i)) &= 6h^\vee \frac{h^\vee+1}{h^\vee+6} - \frac{(5h^\vee-6)(1+h^\vee)}{6+h^\vee} \\ &= h^\vee + 1.\end{aligned}$$

The statement now follows from the equality $(h^\vee+1)\,\mathrm{id}_g = S(\sum_i X_iY_i)$. ■

3.1.1 The algebra $A(\mathfrak{sl}_3)$

The following example is sketched in [CG21] and was further explained to me by Maurice Chayet in an email correspondence.

Example 3.1.2. Chayet and Garibaldi leverage the previous proposition to get a nice description of $A(\mathfrak{sl}_3)$. In this case, σ will also be surjective. After applying the map σ, the resulting multiplication $\star$ is given by

$$P \star Q = [\frac{1}{2}\varepsilon(P \bullet Q) - \frac{3}{2}\varepsilon(P)\varepsilon(Q)]I + \varepsilon(P)Q + \varepsilon(Q)P.$$

Proof. The Lie algebra $\mathfrak{sl}_3$ has type A_2, so in this case the natural representation V is 3-dimensional, and $h^\vee = 3$. Recall that we can construct $\mathfrak{sl}_3$ as the 3-dimensional matrices of trace zero, and the natural 3-dimensional representation is the action of these matrices on a 3-dimensional vector space.

By using Lemma 1.6.7, we can replace the Killing form by the trace form:

$$\mathrm{Tr}(XY) = \frac{1}{6}K(X, Y).$$

Formula 3.1 becomes

$$\sigma(S(XY)) = 18X \bullet Y - 3\,\mathrm{Tr}(XY).$$

Now, by Corollary 2.6.2, we know $A(\mathfrak{sl}_3)$ is exactly 9-dimensional. By dimension count, the injective map $\sigma\colon A(\mathfrak{sl}_3) \to M_3(k)$ has to be an isomorphism. The only thing that remains is the tricky part, namely identifying the multiplication.

We define $\sigma(S(AB)) \star \sigma(S(CD)) \coloneqq \sigma(S(AB) \diamond S(CD))$.

Recall that any matrix can be written as the sum of rank 1 matrices, i.e. matrices of the form $P = ab^\top$, where a and b are 3-dimensional column vectors[1]. For any such matrix P, we have that $P^2 = ab^\top ab^\top = (b^\top a)ab^\top = \mathrm{Tr}(P)P$.

Denote by $\varepsilon(\cdot) = \frac{\mathrm{Tr}(\cdot)}{3}$ the counit of M_3. Observe that, denoting ϵ to be the counit of $A(\mathfrak{g})$, we have $\epsilon = \varepsilon \circ \sigma$.

For such a matrix P, we have that $X = P - \varepsilon(P)$ is in $\mathfrak{sl}_3$. Computing $\sigma(S(X^2))$, we get that

$$\begin{aligned}\sigma(S(X^2)) &= 18\,(P-\varepsilon(P))^2 - 3\,\mathrm{Tr}\left((P-\varepsilon(P))^2\right) \\ &= 18\left(\mathrm{Tr}(P)P - 2\varepsilon(P)P + \varepsilon(P)^2\right) - 3\,\mathrm{Tr}\left(\varepsilon(P)P + \varepsilon(P)^2\right) \\ &= 18\varepsilon(P)P + 18\varepsilon(P)^2 - 9\varepsilon(P)^2 - 9\varepsilon(P)^2 \\ &= 18\varepsilon(P)P.\end{aligned}$$

[1] More formally, there is an isomorphism $M_3(k) \cong V \otimes V$.

Now, all that remains is to figure out how for two rank 1 matrices P and Q, the preimages $X^2 = (P-\varepsilon(P))^2$ and $Y^2 = (Q-\varepsilon(Q))^2$ multiply. Computing the multiplication $\star$, we get

$$\begin{aligned}
\sigma(S(X^2)) \star \sigma(S(Y^2)) &= 3\sigma(S((\operatorname{ad} X)^2 Y, Y)) + 3\sigma(S((\operatorname{ad} Y)^2 X, X)) \\
&\quad + 3\sigma(S([X,Y],[X,Y])) \\
&\quad + K(X,Y)\sigma(S(X,Y)) \\
&= 54\left((\operatorname{ad} X)^2 Y \bullet Y + (\operatorname{ad} Y)^2 X \bullet X + [X,Y]^2\right) && (3.2) \\
&\quad - 9\left(\operatorname{Tr}\left((\operatorname{ad} X)^2 Y \cdot Y\right) + \operatorname{Tr}\left((\operatorname{ad} Y)^2 X \cdot X\right) + \operatorname{Tr}\left([X,Y]^2\right)\right) && (3.3) \\
&\quad + 6\operatorname{Tr}(XY)\left(18 X \bullet Y - 3\operatorname{Tr}(XY)\right). && (3.4)
\end{aligned}$$

Since the trace form is equal to the Killing form up to a scalar, the trace form is also $\mathfrak{g}$-equivariant (this is also easy to show directly). Using this, we can rewrite 3.3 into

$$\begin{aligned}
&\operatorname{Tr}\left((\operatorname{ad} X)^2 Y \cdot Y\right) + \operatorname{Tr}\left((\operatorname{ad} Y)^2 X \cdot X\right) + \operatorname{Tr}\left([X,Y]^2\right) \\
&= -\operatorname{Tr}\left((\operatorname{ad} X)Y \cdot (\operatorname{ad} X)Y\right) + \operatorname{Tr}\left((\operatorname{ad} Y)X \cdot (\operatorname{ad} Y)X\right) + \operatorname{Tr}\left([X,Y]^2\right) \\
&= -\operatorname{Tr}\left([X,Y]^2\right).
\end{aligned}$$

The terms 3.2 are a little more tedious to simplify, but it is a very routine computation.

First we write out $(\operatorname{ad} X)^2 Y \bullet Y$ separately to get

$$\begin{aligned}
(\operatorname{ad} X)^2 Y \bullet Y &= ([X,[X,Y]]) \bullet Y \\
&= ([X, XY - YX]) \bullet Y \\
&= \left(X^2 Y - 2XYX + YX^2\right) \bullet Y \\
&= \frac{1}{2} X^2 Y^2 - XYXY + YX^2 Y - YXYX + \frac{1}{2} Y^2 X^2.
\end{aligned}$$

Of course, by switching the roles of X and Y we get a similar formula for the second term in 3.2. Thus, 3.2 becomes

$$\begin{aligned}
&(\operatorname{ad} X)^2 Y \bullet Y + (\operatorname{ad} Y)^2 X \bullet X + [X,Y]^2 \\
&= X^2Y^2 - 2XYXY + YX^2Y + XY^2X - 2YXYX + Y^2X^2 + (XY - YX)^2 \\
&= X^2Y^2 + Y^2X^2 - (XY)^2 - (YX)^2.
\end{aligned}$$

So eventually we get that

$$\begin{aligned}
324\varepsilon(P)\varepsilon(Q) P \star Q &= 54\left(X^2Y^2 + Y^2X^2 - (XY)^2 - (YX)^2\right) && (3.5) \\
&\quad + 108\operatorname{Tr}(XY) X \bullet Y && (3.6) \\
&\quad + 9\left(\operatorname{Tr}\left([X,Y]^2\right) - 2\operatorname{Tr}(XY)^2\right). && (3.7)
\end{aligned}$$

We will again simplify these three terms separately. First we tackle 3.5. To do this we note that, writing $P = ab^\top, Q = cd^\top$, we obtain

$$\begin{aligned}
PQP &= ab^\top cd^\top ab^\top \\
&= (b^\top c)(d^\top a) ab^\top \\
&= \operatorname{Tr}(PQ)P \\
&= 3\varepsilon(PQ)P.
\end{aligned}$$

Similarly, we have $QPQ = 3\varepsilon(QP)Q = 3\varepsilon(PQ)Q$.

Now comes the main simplification. For 3.5 we get

$$\begin{aligned}
&X^2Y^2 + Y^2X^2 - (XY)^2 - (YX)^2 \\
&= (P-\varepsilon(P))^2 (Q-\varepsilon(Q))^2 + (Q-\varepsilon(Q))^2 (P-\varepsilon(P))^2 \\
&\quad - ((P-\varepsilon(P))(Q-\varepsilon(Q)))^2 - ((Q-\varepsilon(Q))(P-\varepsilon(P)))^2 \\
&= \left(\varepsilon(P)P + \varepsilon(P)^2\right)\left(\varepsilon(Q)Q + \varepsilon(Q)^2\right) + \left(\varepsilon(Q)Q + \varepsilon(Q)^2\right)\left(\varepsilon(P)P + \varepsilon(P)^2\right) \\
&\quad - (PQ - \varepsilon(P)Q - \varepsilon(Q)P + \varepsilon(P)\varepsilon(Q))^2 - (QP - \varepsilon(Q)P - \varepsilon(P)Q + \varepsilon(P)\varepsilon(Q))^2 \\
&= \varepsilon(P)\varepsilon(Q)(PQ+QP) + 2\varepsilon(P)^2\varepsilon(Q)Q + 2\varepsilon(P)\varepsilon(Q)^2P \\
&\quad - 3\varepsilon(PQ)(PQ+QP) + 6\varepsilon(P)\varepsilon(Q)(PQ+QP) + 6\varepsilon(PQ)\varepsilon(Q)P + 6\varepsilon(PQ)\varepsilon(P)Q \\
&\quad - 2\varepsilon(P)\varepsilon(Q)(PQ+QP) - 6\varepsilon(P)^2\varepsilon(Q)Q - 6\varepsilon(P)\varepsilon(Q)^2P - 2\varepsilon(P)\varepsilon(Q)(PQ+QP) \\
&\quad + 4\varepsilon(P)^2\varepsilon(Q)Q + 4\varepsilon(P)\varepsilon(Q)^2P \\
&= (6\varepsilon(P)\varepsilon(Q) - 6\varepsilon(PQ))\, P \bullet Q + 6\varepsilon(PQ)\varepsilon(P)Q + 6\varepsilon(PQ)\varepsilon(Q)P.
\end{aligned}$$

Next, we try to simplify 3.6:

$$\begin{aligned}
\mathrm{Tr}(XY)X \bullet Y &= \mathrm{Tr}\left((P-\varepsilon(P))(Q-\varepsilon(Q))\right)(P \bullet Q - \varepsilon(P)Q - \varepsilon(Q)P + \varepsilon(P)\varepsilon(Q)) \\
&= 3\left(\varepsilon(PQ) - \varepsilon(P)\varepsilon(Q)\right)(P \bullet Q - \varepsilon(P)Q - \varepsilon(Q)P + \varepsilon(P)\varepsilon(Q)) \\
&= 3\left(\varepsilon(PQ) - \varepsilon(P)\varepsilon(Q)\right) P \bullet Q \\
&\quad + 3\left(\varepsilon(P)^2\varepsilon(Q) - \varepsilon(PQ)\varepsilon(P)\right) Q \\
&\quad + 3\left(\varepsilon(P)\varepsilon(Q)^2 - \varepsilon(PQ)\varepsilon(Q)\right) P \\
&\quad + 3\left(\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) - \varepsilon(P)^2\varepsilon(Q)^2\right).
\end{aligned}$$

Finally, we obtain a formula for 3.7:

$$\begin{aligned}
3\varepsilon\left([X,Y]^2\right) - 6\varepsilon(XY)^2 &= 3\varepsilon\left([P,Q]^2\right) - 6\left(\varepsilon(PQ) - \varepsilon(P)\varepsilon(Q)\right)^2 \\
&= 3\varepsilon\left(PQPQ - PQ^2P - QP^2Q + QPQP\right) \\
&\quad - 18\left(\varepsilon(PQ)^2 - 2\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) + \varepsilon(P)^2\varepsilon(Q)^2\right) \\
&= 18\varepsilon(PQ)^2 - 54\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) \\
&\quad - 18\left(\varepsilon(PQ)^2 - 2\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) + \varepsilon(P)^2\varepsilon(Q)^2\right) \\
&= -18\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) - 18\varepsilon(P)^2\varepsilon(Q)^2.
\end{aligned}$$

We can combine this information to obtain (after dividing both sides by 9)

$$\begin{aligned}
36\varepsilon(P)\varepsilon(Q)P \star Q &= 36\varepsilon(P)\varepsilon(Q)P \bullet Q - 36\varepsilon(PQ)P \bullet Q + 36\varepsilon(PQ)\varepsilon(P)Q + 36\varepsilon(PQ)\varepsilon(Q)P \\
&\quad + 36\varepsilon(PQ)P \bullet Q - 36\varepsilon(P)\varepsilon(Q)P \bullet Q \\
&\quad + 36\varepsilon(P)^2\varepsilon(Q)Q - 36\varepsilon(PQ)\varepsilon(P)Q + 36\varepsilon(P)\varepsilon(Q)^2P - 36\varepsilon(PQ)\varepsilon(Q)P \\
&\quad + 36\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) - 36\varepsilon\left(P\right)^2\varepsilon(Q)^2 \\
&\quad - 18\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) - 18\varepsilon(P)^2\varepsilon(Q)^2 \\
&= 36\varepsilon(P)^2\varepsilon(Q)Q + 36\varepsilon(P)\varepsilon(Q)^2P \\
&\quad + 18\varepsilon(PQ)\varepsilon(P)\varepsilon(Q) - 54\varepsilon(P)^2\varepsilon(Q)^2.
\end{aligned}$$

Dividing both sides by $36\varepsilon(P)\varepsilon(Q)$, we get

$$P \star Q = [\frac{1}{2}\varepsilon(P \bullet Q) - \frac{3}{2}\varepsilon(P)\varepsilon(Q)]I + \varepsilon(P)Q + \varepsilon(Q)P.$$

Since this holds for any two rank 1 matrices, it also holds for general matrices. ∎

In the rest of this chapter, we will try to do something similar for the smallest exceptional group, G_2.

3.2 The octonion algebra

It is a well known fact that the 7-dimensional representation of G_2 arises naturally from the theory of *composition algebras*, and more in particular the octonion algebras. That is why we will outline some results about these objects that will be of use later, to determine what the constructed algebra of type G_2 looks like.

Though octonion algebras can be defined over any characteristic, we restrict ourselves to $\operatorname{char} k \neq 2$, as this allows for a uniform treatment, and we have no use for the case $\operatorname{char} k = 2$. The treatment given in this section is based on [SV00]. First we recall the definition of quadratic forms.

Definition 3.2.1 (Quadratic form). A *quadratic form* on a vector space V over a field k is a map $N\colon V \to k$ satisfying

1. $N(\lambda x) = \lambda^2 N(x)$ for all $\lambda \in k$ and $x \in V$,
2. The map $\langle \cdot, \cdot \rangle$ defined by

$$\langle a, b \rangle \coloneqq N(a+b) - N(a) - N(b)$$

is a symmetric bilinear form.

We will call the quadratic form N nondegenerate if the bilinear form $\langle \cdot, \cdot \rangle$ is.

Definition 3.2.2 (Composition algebra). A *composition algebra* is a (not necessarily associative) k-algebra A equipped with a nondegenerate quadratic form $N\colon A \to k$ such that

$$N(ab) = N(a)N(b)$$

for all $a, b \in A$. We will also call N the *norm* of the composition algebra.

This norm can be seen (just as for the real or complex numbers) as the “size squared” of the element we work with. Whenever we talk about orthogonality, then we understand this to be the orthogonality with respect to the associated bilinear form $\langle \cdot, \cdot \rangle$.

Proposition 3.2.3. *Every element x of a composition algebra A with identity e satisfies*

$$x^2 + \langle x, e \rangle x + N(x)e = 0.$$

Moreover, for $x, y \in A$, we have that

$$xy + yx - \langle x, e \rangle y - \langle y, e \rangle x + \langle x, y \rangle e = 0.$$

Proof. See [SV00, Proposition 1.2.3]. ■

It is easy to see that the complex numbers form a real composition algebra of dimension two. Unsurprisingly, we also have an analogue of the conjugation on these algebras.

Definition 3.2.4. Let A be a composition algebra with identity e. We define the *standard involution* $\overline{\cdot}\colon A \to A$ by

$$\overline{x} = \langle x, e \rangle e - x$$

for all $x \in A$.

Lemma 3.2.5. *The standard involution on a composition algebra is an anti-automorphism, i.e. it reverses the algebra multiplication and preserves the norm.*

Proof. It is clearly a linear map. Moreover for $x, y \in A$ we have

$$\begin{aligned}\overline{x} \cdot \overline{y} &= (\langle x, e\rangle e - x)(\langle y, e\rangle e - y) \\ &= \langle x, e\rangle\langle y, e\rangle e - \langle x, e\rangle y - \langle y, e\rangle x + xy \\ &= \langle x, e\rangle\langle y, e\rangle e - \langle x, y\rangle e - yx.\end{aligned}$$

We calculate this first term separately to get

$$\begin{aligned}\langle x, e\rangle\langle y, e\rangle &= (N(x+e) - N(x) - N(e))\,(N(y+e) - N(y) - N(e)) \\ &= N(xy + x + y + e) - N(xy + x) - N(y + e) \\ &- N(xy + y) + N(xy) + N(y) - N(x + e) + N(x) + N(e) \\ &= (\langle xy + x, y + e\rangle - \langle xy, y\rangle - \langle x, e\rangle) \\ &= \langle xy, e\rangle + \langle x, y\rangle.\end{aligned}$$

Plugging this into the first equation, we see that $\overline{x} \cdot \overline{y} = \overline{yx}$.

Next we prove that it preserves the norm structure:

$$\begin{aligned}\langle \overline{x}, \overline{y}\rangle &= \langle \langle x, e\rangle e - x, \langle y, e\rangle e - y\rangle \\ &= 2\langle x, e\rangle\langle y, e\rangle - 2\langle x, e\rangle\langle y, e\rangle + \langle x, y\rangle \\ &= \langle x, y\rangle.\end{aligned}$$

■

To reach our goal of an alternate description of $A(\mathfrak{g}_2)$, we will need certain identities satisfied by composition algebras. For proofs I will refer to [SV00].

Lemma 3.2.6. *for $x, y, z \in A$ we have*

$$\begin{aligned}\langle xy, z\rangle &= \langle y, \overline{x}z\rangle, \\ \langle xy, z\rangle &= \langle x, z\overline{y}\rangle, \\ \langle xy, \overline{z}\rangle &= \langle yz, \overline{x}\rangle.\end{aligned}$$

Proof. See [SV00, Lemma 1.3.2]. ■

Lemma 3.2.7. *For all $x, y \in A$ we have*

$$\begin{aligned}x(\overline{x}y) &= N(x)y, \\ (x\overline{y})y &= N(y)x.\end{aligned}$$

Proof. See [SV00, Lemma 1.3.3]. ■

The Moufang identities are intrinsic to the theory of composition algebras, though we will only need them once or twice.

Proposition 3.2.8 (Moufang identities). *For all $a, x, y \in A$ we have*

$$\begin{aligned}(ax)(ya) &= a((xy)a), \\ a(x(ay)) &= (a(xa))y, \\ x(a(ya)) &= ((xa)y)a.\end{aligned}$$

Proof. See [SV00, Proposition 1.4.1]. ■

These Moufang identities have a very important consequence, namely the alternative laws.

Proposition 3.2.9 (Alternative laws). *For $x, y \in A$ we have*

$$(xy)x = x(yx),$$
$$x(xy) = x^2y,$$
$$(xy)y = xy^2.$$

Proof. See [SV00, Lemma 1.4.2]. ■

The nomenclature here is important, because the identities imply that composition algebras are so-called *alternative* algebras.

Definition 3.2.10. Let A be an algebra over a field k.

- We define the *associator* of three elements $x, y, z \in A$ to be

$$\{x, y, z\} = (xy)z - x(yz).$$

- A will be called an *alternative algebra* if for all $x_1, x_2, x_3 \in A$ and $\pi \in \mathcal{S}_3$ the following holds:

$$\{x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}\} = \operatorname{sgn}(\pi)\{x_1, x_2, x_3\}.$$

With these definitions and the alternative laws, we have the following corollary.

Corollary 3.2.11. *Any composition algebra is an alternative algebra.*

The reason we are interested in composition algebras is for their most interesting examples.

Theorem 3.2.12. *Let A be a composition algebra over a field k. Then one of the following occurs:*

1. $A = k$,
2. $\dim A = 2$*, and then the algebra is a separable quadratic extension of the field* k*,*
3. $\dim A = 4$*, and then the algebra is a non-commutative, associative algebra, called a quaternion algebra,*
4. $\dim A = 8$*, and then the algebra is a non-commutative, non-associative algebra, called an octonion algebra.*

Proof. See [SV00, Theorem 1.6.2]. ■

This theorem is proven through the technique of *doubling*, by which we can essentially stack two copies of an associative composition algebra onto each other, and this would form another composition algebra. Not only that, any composition algebra arises in this way.

At this point we will take a closer look at the octonion algebras, as these are the objects of interest. It turns out that the automorphism group of an octonion algebra is a group of type G_2.

Theorem 3.2.13. *The automorphism group of an octonion algebra defined over a field k is a connected, simple algebraic group of type G_2.*

Proof. See [SV00, Theorem 2.3.5]. ■

To efficiently compute certain things in Chapter 4, we will make use of a standard basis.

Proposition 3.2.14. *Any octonion algebra $\mathbb{O}$ over a field k with* $\operatorname{char} k \neq 2$ *has an orthogonal basis of the form* $e, e_1 = a, e_2 = b, e_3 = ab, e_4 = c, e_5 = ac, e_6 = bc, e_7 = (ab)c$*, with* $N(a)N(b)N(c) \neq 0$.

Proof. See [SV00, Corollary 1.6.3]. ■

Remark 3.2.15. In case k is algebraically closed, we can assume $N(a) = N(b) = N(c) = 1$, and we call such a basis a *standard basis*.[2] For a basis of this form, we can encode the multiplication using the so-called Fano Plane mnemonic (see Figure 3.1). To prove this Fano Plan mnemonic is correct, it suffices to use the Moufang identities and Lemma 3.2.7. It is useful to keep this mnemonic in mind when doing the computations in the next chapter.

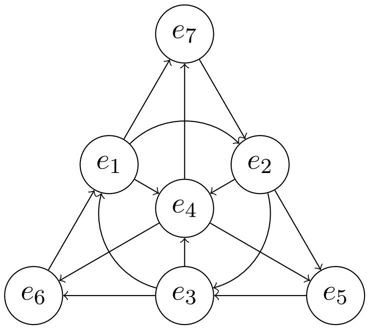


Figure 3.1: The Fano plane mnemonic. If one follows the arrows when multiplying, then the outcome is equal to the third point in the line. Otherwise it is equal minus the third point on the line, e.g. $e_6e_2 = e_4$.

From the viewpoint of the algebraic group, it is more natural to work with the 7-dimensional irreducible representation $W = e^{\perp}$, which we will also call the *(purely) imaginary* octonions. To do this, we need to modify the octonion multiplication to a multiplication on W.

Definition 3.2.16. We will define the *Maltsev product* on the imaginary octonions by

$$a * b := ab - ba \quad \text{for all } a, b \in W.$$

Lemma 3.2.17. *The product $*$ on the imaginary octonions is anticommutative.*

It is easy to see that Lemma 3.2.6 extends to the Maltsev product, in the following way.

Lemma 3.2.18. *For $x, y, z \in W$ imaginary octonions, we have that*

$$\langle x * y, z\rangle = -\langle x, z * y\rangle.$$

The following is a rather technical result, but it will simplify a lot of the computations in the next section.

Lemma 3.2.19.

- *If $a, b \in W$ are imaginary octonions and $x \perp a, b$, then*

$$\{a, x, b\} = \frac{x * (a * b)}{2}.$$

- *If $a, b, x \in W$ are imaginary octonions, then*

$$(x * a) * b + (x * b) * a = 2\langle a, x\rangle b + 2\langle b, x\rangle a - 4\langle a, b\rangle x.$$

[2] As if there are not enough "standard" things.

Proof. • The second equation in Lemma 3.2.3 gives us a clue about when the normal product anticommutes, namely if all involved elements are orthogonal. We make use of this property. First consider the case where a and b are anisotropic. We can without loss of generality consider $a \perp b$, since orthogonally projecting b does not change either the left hand side or the right hand side. We first consider the case $x \perp ab = a * b$:

$$\begin{aligned}\{a,x,b\} &= (ax)b - a(xb)\\ &= -(xa)b + a(bx)\\ &= -\{x,a,b\} - x(ab) - \{a,b,x\} + (ab)x\\ &= \{a,x,b\} + \{a,x,b\} - x(a*b)\\ &= \{a,x,b\} + \{a,x,b\} - \frac{x(a*b) - (a*b)x}{2}\\ &= 2\{a,x,b\} - \frac{x*(a*b)}{2}.\end{aligned}$$

Now we treat the case $x = ab$:

$$\begin{aligned}\{a,ab,b\} &= a((ab)b) - (a(ab))b\\ &= N(a)N(b) - N(a)N(b)\\ &= 0.\end{aligned}$$

Since a, b are anisotropic,ab is also anisotropic. This means that we have verified the identity $\{a,x,b\} = \frac{x*(a*b)}{2}$ where x ranges over a basis of W. By linearity, the identity thus holds for all x.

Now, since the identity holds for anisotropic a, b, it will hold for more general a, b as well due to the linearity of the expression.

- We will prove this by viewing both sides as maps with argument x. As in the previous point, we can assume without loss of generality that a and b are anisotropic.

  First, considering the case $x = a$ we have

$$\begin{aligned}(a*a)*b + (a*b)*a &= -(b*a)*a\\ &= -(ba - ab)*a\\ &= -(ba)a + a(ba) + (ab)a - a(ab)\\ &= 2N(a)b - (ba)a + \langle a,e\rangle ba + \langle ba,e\rangle a - \langle a,ba\rangle e\\ &\quad - a(ab) + \langle ab,e\rangle a + \langle a,e\rangle ab - \langle ab,a\rangle e\\ &= 4N(a)b - 2\langle a,b\rangle a.\end{aligned}$$

  By symmetry, we obtain something similar for b. In case $x \perp a, b$ we get

$$\begin{aligned}(x*a)*b + (x*b)*a &= (xa - ax)*b + (xb - bx)*a\\ &= 2(xa)b - 2b(xa) + 2(xb)a - 2a(xb)\\ &= 2\{x,a,b\} + 2x(ab) - 2\{b,a,x\} + 2(ba)x\\ &\quad + 2\{x,b,a\} + 2x(ba) - 2\{a,b,x\} + 2(ab)x\\ &= 2x(ab+ba) + 2(ab+ba)x\\ &= 2x(\langle a,e\rangle b + \langle b,e\rangle a - \langle a,b\rangle e) + 2(\langle a,e\rangle b + \langle b,e\rangle a - \langle a,b\rangle e)x\\ &= -4\langle a,b\rangle x.\end{aligned}$$

  Combining all cases, we get the formula we want. ∎

Remark 3.2.20. Compare the second point of this lemma to [SV00, Lemma 1.3.1].

In the next subsection, we will discuss certain *derivations* of the octonion algebra. Theorem 3.2.23 explains why.

Definition 3.2.21. Let A be a k-algebra, and $D\colon A \to A$ a linear map. We call D a derivation if it satisfies the Leibniz rule, i.e.

$$D(xy) = D(x)y + xD(y), \text{ for all } x, y \in A.$$

These derivations form a Lie algebra structure.

Lemma 3.2.22. *Let A be a k-algebra. The set of derivations $D\colon A \to A$ is a Lie algebra with respect to the commutator product, i.e.*

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \text{ for all derivations } D_1, D_2.$$

Proof. We have to show that for two derivations D_1, D_2 the commutator is also a derivation. We check that it satisfies the Leibniz rule.

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(xy) &= D_1(D_2(x)y + xD_2(y)) - D_2(D_1(x)y + xD_1(y)) \\ &= (D_1 \circ D_2)(x)y + x(D_1 \circ D_2)(y) + D_1(x)D_2(y) + D_2(x)D_1(y) \\ &\quad - (D_2 \circ D_1)(x)y - x(D_1 \circ D_2)(y) - D_1(x)D_2(y) - D_2(x)D_1(y) \\ &= (D_1 \circ D_2 - D_2 \circ D_1)(x)y + x(D_1 \circ D_2 - D_2 \circ D_1)(y)\end{aligned}$$

This proves that the commutator product on the derivations is well defined. ∎

Theorem 3.2.23. *The derivation algebra of an octonion algebra $\mathbb{O}$ is isomorphic to the Lie algebra of its automorphism group.*

Proof. See [SV00, Proposition 2.4.5]. ∎

3.2.1 Standard derivations

In this section, we will use so called *standard derivations*, introduced by R. D. Schafer and studied in [Sch95], to make sense of the embedding given in [CG21]. In this section, we will denote an octonion algebra over k by $\mathbb{O}$.

Definition 3.2.24. Let $a, b \in \mathbb{O}$ be octonions. We define the standard derivation $D_{a,b}$ by

$$D_{a,b} := [L_a, L_b] + [L_a, R_b] + [R_a, R_b],$$

where L_x (respectively R_x) stands for left (respectively right) multiplication by x for all $x \in \mathbb{O}$.

Lemma 3.2.25. *Let $a, b \in \mathbb{O}$ be octonions. The following hold:*

1. *$D_{a,b}$ is a derivation,*
2. *$D_{a,b} = -D_{b,a}$,*
3. *$D_{a,a} = 0$,*
4. *$D_{a,b} = 0$ if $a \in k$.*

Proof. 1. This is proven in [Sch95, Identity (3.70), p.77].

2. This follows immediately from the fact that the octonions are alternative (Corollary 3.2.11).

3. All commutators in the definition become zero.

4. All commutators in the definition become zero. ■

The reason we want to use these derivations is because they describe the commutator product explicitly in terms of the underlying octonion algebra.

Lemma 3.2.26. *Let D be a derivation of $\mathbb{O}$, and $a, b \in \mathbb{O}$. Then we have*

$$[D, D_{a,b}] = D_{Da,b} + D_{a,Db}.$$

Proof. The definition of a derivation is equivalent to the relation

$$[D, L_x] = L_{D(x)}$$

and equivalent to

$$[D, R_x] = R_{D(x)}.$$

Then, we compute $[D, D_{a,b}]$ using the Jacobi identity:

$$\begin{aligned}
[D, D_{a,b}] &= [D, [L_a, L_b] + [L_a, R_b] + [R_a, R_b]] \\
&= [D, [L_a, L_b]] + [D, [L_a, R_b]] + [D, [R_a, R_b]] \\
&= -[L_a, [L_b, D]] - [L_b, [D, L_a]] \\
&\quad - [L_a, [R_b, D]] - [R_b, [D, L_a]] \\
&\quad - [R_a, [R_b, D]] - [R_b, [D, R_a]] \\
&= [L_a, L_{D(b)}] + [L_{D(a)}, L_b] \\
&\quad + [L_a, R_{D(b)}] + [L_{D(a)}, R_b] \\
&\quad + [R_a, R_{D(b)}] + [R_{D(a)}, R_b] \\
&= D_{Da,b} + D_{a,Db}.
\end{aligned}$$

This proves the lemma. ■

There is some linear dependency between these derivations, which we will use in Chapter 4.

Lemma 3.2.27. *For $a, b, c \in \mathbb{O}$ we have*

$$D_{ab,c} + D_{ca,b} + D_{bc,a} = 0.$$

Proof. This is [Sch95, Identity (3.73)]. ■

To find a formula for the embedding σ, we try to give a formula for the derivations in terms of the bilinear form on the octonions.

Proposition 3.2.28. *Let $a, b \in W$ be imaginary anisotropic octonions. We have the following:*

1. $D_{a,b}(a) = 4N(a)b - 2\langle a, b\rangle a$,

2. $D_{a,b}(b) = -4N(b)a + 2\langle a, b\rangle b$,

3. $D_{a,b}(ab) = 0$,

4. $D_{a,b}(x) = \frac{x*(a*b)}{2}$ *for all x with $x \perp a, b, ab$.*

Proof. 1. This is a straightforward calculation:

$$\begin{aligned}
D_{a,b}(a) &= a(ba) - b(aa) + a(ab) - (aa)b + (ab)a - (aa)b \\
&= a(ba) + bN(a) - N(a)b + N(a)b + (ab)a + N(a)b \\
&= a(-ab + \langle b, e\rangle a + \langle a, e\rangle b - \langle a, b\rangle e) \\
&\quad + (-ba + \langle a, e\rangle b + \langle b, e\rangle a - \langle b, a\rangle e)a \\
&\quad + 2N(a)b \\
&= 4N(a)b - 2\langle a, b\rangle.
\end{aligned}$$

2. This is (anti)symmetrical to the first case and does not require an additional proof.

3. We prove that in fact, $D_{a,b}(ab) = 0$:

$$\begin{aligned}
D_{a,b}(ab) &= a(b(ab)) - b(a(ab)) + a((ab)b) - (a(ab))b + ((ab)b)a - ((ab)a)b \\
&= a(b(ab)) - ((ab)a)b \\
&\quad - N(a)N(b) + N(a)N(b) - N(a)N(b) + N(a)N(b) \\
&= a(b(-ba - \langle b, a\rangle e)) - ((-ba - \langle b, a\rangle e)a)b \\
&= -N(b)N(a) - \langle b, a\rangle ab + N(b)N(a) + \langle b, a\rangle ab \\
&= 0.
\end{aligned}$$

4. The condition that $x \perp a, b, ab$ ensures the elements anticommute (with respect to the normal octonion product) by Lemma 3.2.3. Armed with this knowledge, we compute $D_{a,b}(x)$. Note also that $xb, xa \perp a, b$ because a, b are anisotropic and all of them are purely imaginary:

$$\begin{aligned}
D_{a,b}(x) &= a(bx) - b(ax) + a(xb) - (ax)b + (xb)a - (xa)b \\
&= (xb)a - b(ax) \\
&= \{a, x, b\} \\
&= \frac{x * (a * b)}{2}.
\end{aligned}$$

■

Corollary 3.2.29. *For $a, b, x \in W$ imaginary octonions, we have*

$$D_{a,b}(x) = 3b\langle a, x\rangle - 3a\langle b, x\rangle + \frac{x * (a * b)}{2}.$$

Proof. By the previous proposition, this holds when a and b are anisotropic. By Proposition 3.2.14, we have a basis of anisotropic elements. Since the definition of $D_{a,b}$ is linear in both a and b, the corollary holds for general $a, b \in W$. ■

To end this section, we include the result proven by Schafer.

Theorem 3.2.30 ([Sch95]). *Every derivation of the octonions over a field k with $\operatorname{char} k \neq 2, 3$ is a linear combination of standard derivations.*

Proof. See [Sch95, Corollary 3.29]. ■

Remark 3.2.31. This theorem, together with Theorem 3.2.23 shows that we can identify the Lie algebra $\mathfrak{g}_2$ with linear combinations of standard derivations, where the Lie bracket acts as in Lemma 3.2.26.

An important consequence of this fact, which maybe is not highlighted enough in this section, is that for any derivation, its image is contained in $W = e^{\perp}$.

3.3 The brother

In this section, we use the explicit formulas from the previous section for the standard derivations to describe the embedding from Proposition 3.1.1.

Recall that $\mathcal{H}(W)$ stands for the symmetric operators on W with respect to its associated bilinear form.

Lemma 3.3.1. *We have an isomorphism*

$$\begin{aligned} \mathrm{S}^2 W &\to \mathcal{H}(W) \\ ab &\mapsto \frac{a\langle b, \cdot\rangle + b\langle a, \cdot\rangle}{2}. \end{aligned}$$

Moreover, this isomorphism is G_2-equivariant.

Proof. It suffices to observe that for any $\omega \in G_2$, due to the G_2-invariance of the scalar product,

$$\frac{\omega a\langle \omega b, \cdot\rangle + \omega b\langle \omega a, \cdot\rangle}{2} = \omega \circ \left(\frac{a\langle b, \cdot\rangle + b\langle a, \cdot\rangle}{2}\right) \circ \omega^{-1}.$$

By the nondegeneracy of the bilinear form, this map is injective. By dimension count, it is also an isomorphism. ∎

This isomorphism will be very important. For now, it is merely suggestive. To avoid massive expressions, we will write ab as a shorthand for $\frac{a\langle b,\cdot\rangle+b\langle a,\cdot\rangle}{2}$. We will from this point forward also denote the octonion explicitly by $\cdot$ to avoid confusion. Because the embedding in Proposition 3.1.1 is dependent on the Killing form, which by Lemma 1.6.7, is dependent on the trace form, we add a trivial (but nice) lemma.

Lemma 3.3.2. *With the notation from above, we have*

$$\mathrm{Tr}(ab) = \langle a, b\rangle.$$

3.3.1 The relation to the octonions

Using Corollary 3.2.29 and Lemma 3.2.19, we can give an explicit formula for the embeddding in Proposition 3.1.1.

Corollary 3.3.3. *If a, b, c, d are imaginary octonions, then for the image of $S(D_{a,b}D_{c,d})$ under the embedding σ, we find*

$$\begin{aligned} \sigma(S(D_{a,b}D_{c,d})) = &-216\left(\langle a, c\rangle bd - \langle a, d\rangle bc + \langle b, d\rangle ac - \langle b, c\rangle ad\right) \\ &- 36\left((a * (c * d))b - (b * (c * d))a + (c * (a * b))d - (d * (a * b))c\right) \\ &+ 12(a * b)(c * d) \\ &+ 18\left(2\langle a, c\rangle\langle b, d\rangle - 2\langle a, d\rangle\langle b, c\rangle - \langle a * b, c * d\rangle\right) \mathrm{id}_W . \end{aligned}$$

Proof. We compute $D_{a,b} \bullet D_{c,d}$ using Corollary 3.2.29. To reduce the amount of terms involved, we first compute $D_{a,b} \circ D_{c,d}$ and get

$$\begin{aligned} D_{a,b} \circ D_{c,d} = &\left(3b\langle a, \cdot\rangle - 3a\langle b, \cdot\rangle + \frac{1}{2}R^*_{a*b}\right)\left(3d\langle c, \cdot\rangle - 3c\langle d, \cdot\rangle + \frac{1}{2}R^*_{c*d}\right) \\ &= 9b\langle a, d\rangle\langle c, \cdot\rangle - 9b\langle a, c\rangle\langle d, \cdot\rangle \\ &\quad - 9a\langle b, d\rangle\langle c, \cdot\rangle + 9a\langle b, c\rangle\langle d, \cdot\rangle \\ &\quad - \frac{3}{2}b\langle a * (c * d), \cdot\rangle + \frac{3}{2}a\langle b * (c * d), \cdot\rangle \\ &\quad + \frac{3}{2}d * (a * b)\langle c, \cdot\rangle - \frac{3}{2}c * (a * b)\langle d, \cdot\rangle \\ &\quad + \frac{1}{4}R^*_{a*b} \circ R^*_{c*d}, \end{aligned}$$

where we make use of Lemma 3.2.18. By symmetry, we see that

$$\begin{aligned} D_{a,b} \bullet D_{c,d} = &-9\left(\langle a, c\rangle bd - \langle a, d\rangle bc + \langle b, d\rangle ac - \langle b, c\rangle ad\right) \\ &- \frac{3}{2}\left((a * (c * d))b - (b * (c * d))a + (c * (a * b))d - (d * (a * b))c\right) \\ &+ \frac{1}{4}R^*_{a*b} \bullet R^*_{c*d}. \end{aligned}$$

Now we make use of Lemma 3.2.19. Rephrasing the second point into the language of operators, we get

$$R^*_{a*b} \bullet R^*_{c*d} = 2(a * b)(c * d) - 2\langle a * b, c * d\rangle \operatorname{id}_W .$$

To compute the trace of this expression, we take note of Lemma 3.3.2:

$$\begin{aligned} \operatorname{Tr}(D_{a,b} \bullet D_{c,d}) = \operatorname{Tr}(D_{a,b} \circ D_{c,d}) &= -9\left(2\langle a, c\rangle\langle b, d\rangle - 2\langle a, d\rangle\langle b, c\rangle\right) \\ &\quad - \frac{3}{2}\left(\langle a * (c * d), b\rangle - \langle b * (c * d), a\rangle + \langle c * (a * b), d\rangle - \langle d * (a * b), c\rangle\right) \\ &\quad + \frac{1}{2}\left(\langle a * b, c * d\rangle - 7\langle a * b, c * d\rangle\right) \\ &= -18(\langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle) \\ &\quad + 6\langle a * b, c * d\rangle \\ &\quad - 3\langle a * b, c * d\rangle \\ &= -18(\langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle) + 3\langle a * b, c * d\rangle. \end{aligned}$$

We again used Lemma 3.2.18. The corollary then follows from plugging these computations into (first using Lemma 1.6.7)

$$\sigma(S(D_{a,b}D_{c,d})) = 24D_{a,b} \bullet D_{c,d} - 2\operatorname{Tr}(D_{a,b}D_{c,d}) \operatorname{id}_W . \qquad \blacksquare$$

This allows us to see the following.

Proposition 3.3.4. *The map*

$$\sigma \colon A(\mathfrak{g}_2) \to \mathcal{H}(W)$$

is an isomorphism.

Proof. Dimension count, by Corollary 2.6.2. ■

For convenience, we also prove a shorter formula in a special case.

Proposition 3.3.5. *Let $a, b \in W$ be imaginary octonions. Then the embedding σ on $D^2_{a,b}$ is given by*

$$\sigma(S(D^2_{a,b})) = 12\left(-12N(a)bb - 12N(b)aa + (a*b)(a*b) + 12\langle a,b\rangle ab\right).$$

Proof. We have

$$\begin{aligned}\sigma(S(D^2_{a,b})) = &-216\left(\langle a,a\rangle bb - \langle a,b\rangle ab + \langle b,b\rangle aa - \langle a,b\rangle ab\right)\\ &-36\left((a*(a*b))b - (b*(a*b))a + (a*(a*b))b - (b*(a*b))a\right)\\ &+12(a*b)(a*b)\\ &+18\left(2\langle a,a\rangle\langle b,b\rangle - 2\langle a,b\rangle\langle a,b\rangle - \langle a*b, a*b\rangle\right)\mathrm{id}_W.\end{aligned}$$

For the terms on the second line, we note that $ab \perp a$ for two purely imaginary octonions a, b. Thus we have

$$\begin{aligned}a*(a*b) &= a*(ab - ba)\\ &= a(ab-ba) - (ab-ba)a\\ &= 2(a(ab) - a(ba))\\ &= -2N(a)b - 2a(-ab + \langle b,e\rangle a + \langle a,e\rangle b - \langle a,b\rangle e)\\ &= -4N(a)b + 2\langle a,b\rangle a.\end{aligned}$$

By this computation, it follows that

$$\begin{aligned}&(a*(a*b))b - (b*(a*b))a + (a*(a*b))b - (b*(a*b))a\\ &= -8N(a)bb + 4\langle a,b\rangle ab - 8N(b)aa + 4\langle a,b\rangle ab.\end{aligned}$$

For the terms on the fourth line, we get

$$\begin{aligned}&2\langle a,a\rangle\langle b,b\rangle - 2\langle a,b\rangle\langle a,b\rangle - \langle a*b, a*b\rangle\\ &= 2\langle a,a\rangle\langle b,b\rangle - 2\langle a,b\rangle\langle a,b\rangle - \langle 2ab - \langle a,b\rangle e, 2ab - \langle a,b\rangle e\rangle\\ &= 2\langle a,a\rangle\langle b,b\rangle - 2\langle a,b\rangle\langle a,b\rangle - 2\langle a,a\rangle\langle b,b\rangle + 4\langle a,b\rangle\langle a,b\rangle - 2\langle a,b\rangle\langle a,b\rangle\\ &= 0.\end{aligned}$$

Plugging these computations into the original formula, we obtain

$$\begin{aligned}\sigma(S(D^2_{a,b})) &= -432\left(N(a)bb - \langle a,b\rangle ab + N(b)aa\right)\\ &\quad -36\left(-8N(a)bb + 8\langle a,b\rangle ab - 8N(b)aa\right)\\ &\quad +12(a*b)(a*b)\\ &= 12\left(-12N(a)bb - 12N(b)aa + 12\langle a,b\rangle ab + (a*b)(a*b)\right).\end{aligned}$$

■

As another corollary, we get a nice formula for a preimage of very basic elements.

Proposition 3.3.6. *If $a, c \in W$ are orthogonal imaginary octonions, then we have*

$$\sigma(D_{a,a*c}D_{c,a*c}) = -768N(a)N(c)ac.$$

Proof. Bilinearizing Proposition 3.3.5, we get (for arbitrary imaginary octonions)

$$\begin{aligned}\sigma(D_{a,b}D_{c,b}) = 12\,(&-6\langle a,c\rangle bb - 12N(b)ac + 6\langle a,b\rangle cb\\ &+6\langle c,b\rangle ab + (a*b)(c*b))\,.\end{aligned}$$

Now we specify to the case where $a \perp c$ and $b = a*c = 2ac$ hold. Notice that in this case we also have $a \perp ac \perp c$, because both a and c are imaginary octonions. Thus we get

$$\begin{aligned}\sigma(D_{a,a*c}D_{c,a*c}) &= 12\,(-12N(2ac)ac\\ &\quad +(a*(a*c))(c*(a*c)))\\ &= 12(-48N(a)N(c)ac - 16N(a)N(c)ca)\\ &= -768N(a)N(c)ac.\end{aligned}$$

■

Thus we can find nice preimages for certain elements in $\mathrm{S}^2 W$. By using representation theory to our advantage, this is all we will need to determine the product on $\sigma(A(\mathfrak{g}_2))$.

Lemma 3.3.7. *For two imaginary octonions $a, b \in W$ and a standard basis as in Remark 3.2.15, we have*

$$\sum_{i=1}^{7}(e_i*a)(e_i*b) = 2\langle a,b\rangle \sum_{i=1}^{7} e_ie_i - 4ab.$$

Proof. We prove this for $a = e_j, b = e_k$ for certain $j, k \in \{1,\ldots,7\}$. Then the general case follows from bilinearization.

Suppose first $j = k$. Then, because of the way the standard basis is constructed (See Remark 3.2.15), $(e_i * e_j)(e_i * e_j) = 4e_le_l$ for some $l \neq i, j$, if $i \neq j$. Moreover, given j, the index i determines l uniquely. So we have

$$\begin{aligned}\sum_{i=1}^{7}(e_i*e_j)(e_i*e_j) &= 4\sum_{\substack{i=1\\ i\neq j}}^{7} e_ie_i\\ &= 2\langle e_j,e_j\rangle\sum_{i=1}^{7} e_ie_i - 4e_je_j.\end{aligned}$$

Now suppose $j \neq k$. If $i = j$ or k, then $(e_i * e_j)(e_i * e_k) = 0$. Now denote with r the unique index such that $e_r * e_j = \pm 2e_k$.

Then, for all $i \neq r, j, k$, there is a unique i' such that $e_i \cdot e_j = \pm e_{i'} \cdot e_k$.

For this i' we have that $e_{i'} * e_j = \mp e_i * e_k$. Indeed, we compute

$$\begin{aligned}e_{i'}\cdot e_j &= \mp e_{i'}\cdot(e_i\cdot(e_{i'}\cdot e_k))\\ &= \mp(e_{i'}\cdot(e_i\cdot e_{i'}))\cdot e_k\\ &= \pm(e_{i'}\cdot(e_{i'}\cdot e_i))\cdot e_k\\ &= \mp e_i\cdot e_k\end{aligned}$$

by the Moufang identities (Proposition 3.2.8) and the fact that the elements $e_i, i \in \{1,\ldots,7\}$ anti-commute. This proves that in the summation, the index i and index i' will cancel each other.

Using this, we see that

$$
\begin{aligned}
\sum_{i=1}^{7}(e_i * e_j)(e_i * e_k) &= 4(e_r * e_j)(e_r * e_k) \\
&= -4e_j e_k \\
&= 2\langle e_j, e_k\rangle \sum_{i=1}^{7} e_i e_i - 4e_j e_k.
\end{aligned}
$$

In this next theorem, we make use of Lemma 3.3.1.

Theorem 3.3.8. *The algebra $A(\mathfrak{g}_2)$ is isomorphic to the symmetric square of the imaginary octonions $\mathrm{S}^2\, W$, with multiplication given by*

$$
\begin{aligned}
ab \star cd = \frac{1}{12}\left(\langle a,c\rangle bd + \langle a,d\rangle bc + \langle b,c\rangle ad + \langle b,d\rangle ac + \langle a,b\rangle cd + \langle c,d\rangle ab\right) \\
- \frac{1}{48}\left((a*c)(b*d) + (a*d)(b*c)\right).
\end{aligned}
$$

Proof. By Lemma 3.3.1 and Proposition 3.3.4, it only remains to show that

$$\sigma(v) \star \sigma(w) = \sigma(v \diamond w)$$

for all $v, w \in A(\mathfrak{g}_2)$.

In principle, one can use the formulas derived in the section and plug in the numbers to get the description we have. However, one would have to simplify $190+$ terms to 8. We will make use of the abstract backdoor known as representation theory.

In this proof, we can assume without loss of generality that k is algebraically closed. We denote a standard basis as in Remark 3.2.15.

Recall from Proposition 2.6.1 that $A(\mathfrak{g}_2) = k \oplus V$, where V is the irreducible 27-dimensional representation of G_2, and that multiplication is of the form

$$(a,u) \diamond (b,v) = (ab + f(u,v), av + bu + u \times v).$$

For a certain symmetric bilinear form f and symmetric multiplication $\times$ on V. We have to determine both f and $\times$. We observe that under the embedding σ, V gets sent to the subspace of trace zero elements, i.e.

$$\sigma(V) = \left\{ \sum_i \lambda_i a_i b_i \,\middle|\, \sum_i \langle a_i, b_i\rangle = 0 \right\}.$$

It is known that the space of bilinear symmetric products on the 27-dimensional representation of G_2 is exactly 2-dimensional, see Remark 1.4.14. Two linearly independent products on $\sigma(V)$ are

$$ab \diamond_1 cd = (a*c)(b*d) + (a*d)(b*c) - \frac{1}{14}(\langle a*c, b*d\rangle + \langle a*d, b*c\rangle)I,$$

and

$$ab \diamond_2 cd = \langle a,c\rangle bd + \langle a,d\rangle bc + \langle b,c\rangle ad + \langle b,d\rangle ac - \frac{2}{14}(\langle a,c\rangle\langle b,d\rangle + \langle a,d\rangle\langle b,c\rangle)I,$$

where $I := \sum_{i=1}^{7} e_i e_i$. It can easily be seen that these are linearly independent, since $e_1e_2 \diamond_1 e_1e_2 = -4e_3e_3 + \frac{4I}{7}$ and $e_1e_2 \diamond_2 e_1e_2 = 2e_1e_1 + 2e_2e_2 - \frac{4I}{7}$.

Now we compute $e_1e_2 \star e_1e_2$, using Lemma 3.2.26, Proposition 3.3.5 and Corollary 3.3.6. Let a, b be orthogonal, anisotropic, imaginary octonions. Then we have

$$\begin{aligned} S(XY) \diamond S(XY) &= 2\left(S(X, (\operatorname{ad} Y \circ \operatorname{ad} X)(Y)) + S(Y, (\operatorname{ad} X \circ \operatorname{ad} Y)(X)) + S([X,Y],[X,Y])\right) \\ &\quad + \frac{1}{4}K(X,X)S(Y^2) + \frac{1}{2}K(X,Y)S(XY) + \frac{1}{4}K(Y,Y)S(X^2). \end{aligned}$$

Next, we compute the involved commutator brackets using Lemma 3.2.26 and Proposition 3.2.29:

$$\begin{aligned} [D_{a,a*b}, D_{b,a*b}] &= D_{b,-16N(a)N(b)a}, \\ [D_{a,a*b}, D_{b,a}] &= D_{b,4N(a)a*b}, \\ [D_{b,a*b}, D_{a,b}] &= D_{a,4N(b)a*b}. \end{aligned}$$

Substituting $X = D_{a,a*b}$ and $Y = D_{b,a*b}$ in the product formula, we get

$$\begin{aligned} S(XY) \diamond S(XY) = {}& 128N(a)N(b)^2 S(D_{a,a*b}^2) + 128N(a)^2N(b)S(D_{b,a*b}^2) \\ & - 512N(a)^2N(b)^2S(D_{a,b}^2) \\ & + \operatorname{Tr}(D_{a,a*b}^2)S(D_{b,a*b}^2) + \operatorname{Tr}(D_{b,a*b}^2)S(D_{a,a*b}^2). \end{aligned}$$

Now we can use Proposition 3.3.5 to get

$$\begin{aligned} \sigma(S(D_{a,b}^2)) &= 12\left(-12N(a)bb - 12N(b)aa + (a*b)(a*b)\right), \\ \sigma(S(D_{a,a*b}^2)) &= 12\left(-48N(a)N(b)aa - 12N(a)(a*b)(a*b) + 16N(a)^2bb\right), \\ \sigma(S(D_{b,a*b}^2)) &= 12\left(-48N(a)N(b)bb - 12N(b)(a*b)(a*b) + 16N(b)^2aa\right). \end{aligned}$$

We also have, by Corollary 3.3.3,

$$\begin{aligned} \operatorname{Tr}(D_{a,a*b}^2)) &= -192N(a)^2N(b), \\ \operatorname{Tr}(D_{b,a*b}^2)) &= -192N(a)N(b)^2. \end{aligned}$$

Plugging all of this information in the product formula, we get for anisotropic orthogonal imaginary octonions a and b,

$$\begin{aligned} 768^2N(a)^2N(b)^2 ab \star ab &= 96 \cdot 4^2 \cdot N(a)^2N(b)^2(64(N(b)aa + N(a)bb) + 8a*ba*b) \\ \iff ab \star ab &= \frac{1}{48}(a*b)(a*b) + \frac{1}{12}(\langle b,b\rangle aa + \langle a,a\rangle bb). \end{aligned}$$

Or, if $a = e_1, b = e_2$, we get $e_1e_2 \star e_1e_2 = \frac{1}{12}e_3e_3 + \frac{1}{6}(e_1e_1 + e_2e_2)$.

We know the multiplication on V should be a linear combination of the products $\diamond_1$ and $\diamond_2$. By looking at the computations for e_1e_2, we determine that $\sigma \circ \times = -\frac{1}{48} \diamond_1 + \frac{1}{12}\diamond_2$. Now, the only thing that remains is to determine the bilinear form f. However, we again know that there is only one symmetric bilinear form (Remark 1.4.14) on the 27-dimensional representation, defined up to a scalar.

The sceptical reader would remark that $\diamond_1$ and $\diamond_2$ give us bilinear forms that seem to be different. But we will have a closer look at the bilinear form arising from $\diamond_2$ right now.

$$\begin{aligned} &\langle a*c, b*d\rangle + \langle a*d, b*c\rangle \\ &= -\langle a, (b*d)*c\rangle - \langle a, (b*c)*d\rangle \\ &= -\langle a, (b*d)*c + (b*c)*d\rangle. \end{aligned}$$

We further manipulate this using Lemma 3.2.19:

$$
\begin{aligned}
&= -\langle a, 2\langle c,b\rangle d + 2\langle d,b\rangle c - 4\langle c,d\rangle b\rangle \\
&= -2\langle a,c\rangle\langle b,d\rangle - 2\langle a,d\rangle\langle b,c\rangle + 4\langle c,d\rangle\langle a,b\rangle.
\end{aligned}
$$

Now, since we only work with the trace zero subspace of the symmetric square, we can drop the last term. We can thus see that

$$f(ab, cd) = \lambda(\langle a,c\rangle\langle b,d\rangle + \langle a,d\rangle\langle b,c\rangle),$$

where λ is a scalar factor, and

$$ab \diamond_1 cd = (a * c)(b * d) + (a * d)(b * c) + \frac{2}{14}(\langle a,c\rangle\langle b,d\rangle + \langle a,d\rangle\langle b,c\rangle)I$$

for traceless elements ab and cd.[3]

Again using the toy computation above, we get that $\lambda = \frac{5}{24 \cdot 7}$.

To reduce this recipe to the nice expression we need one more computation, which is admittedly abusing some notation. Here, Lemma 3.3.7 gives us

$$
\begin{aligned}
I \times ab &= \sum_i \frac{-1}{48}(2(e_i * a)(e_i * b)) - \frac{2}{48 \cdot 14}(2\langle e_i, a\rangle\langle e_i, b\rangle)I \\
&\quad + \frac{1}{12}(2\langle e_i, a\rangle e_i b + 2\langle e_i, b\rangle e_i a) - \frac{2}{12 \cdot 14}(2\langle e_i, a\rangle\langle e_i, b\rangle)I \\
&= -\frac{1}{24}(2\langle a,b\rangle I - 4ab) - \frac{1}{12 \cdot 7}\langle a,b\rangle I + \frac{2}{3}ab - \frac{1}{21}\langle a,b\rangle I \\
&= \frac{5}{6}ab - \frac{1}{7}\langle a,b\rangle I,
\end{aligned}
$$

and

$$f(I, ab) = \frac{5}{6 \cdot 7}\langle a,b\rangle.$$

[3] This expression is not the same when we multiply elements of non zero trace. Some caution is needed when verifying the computations below.

For any $ab, cd \in \mathrm{S}^2 W$ we get

$$
\begin{aligned}
&\left(\frac{\langle a,b\rangle}{14} I + ab - \frac{\langle a,b\rangle}{14} I\right) \star \left(\frac{\langle c,d\rangle}{14} I + cd - \frac{\langle c,d\rangle}{14} I\right) \\
&= \left(\frac{2\langle a,b\rangle\langle c,d\rangle}{14^2} + \frac{1}{2} f\left(ab - \frac{\langle a,b\rangle}{14} I, cd - \frac{\langle c,d\rangle}{14} I\right)\right) I \\
&\quad + \frac{\langle a,b\rangle}{7} cd + \frac{\langle c,d\rangle}{7} ab - \frac{4\langle a,b\rangle\langle c,d\rangle}{14^2} I - \frac{\langle c,d\rangle}{14} I \times ab - \frac{\langle a,b\rangle}{14} I \times cd \\
&\quad + ab \times cd + \frac{\langle a,b\rangle\langle c,d\rangle}{14^2} I \times I \\
&= \left(-\frac{2\langle a,b\rangle\langle c,d\rangle}{14^2} + \frac{5}{48\cdot 7}\left(\langle a,c\rangle\langle b,d\rangle + \langle a,d\rangle\langle b,c\rangle - \frac{2\langle a,b\rangle\langle c,d\rangle}{7}\right)\right) I \\
&\quad + \frac{\langle a,b\rangle}{7} cd + \frac{\langle c,d\rangle}{7} ab \\
&\quad - \frac{\langle c,d\rangle}{14}\left(\frac{5}{6} ab - \frac{1}{7}\langle a,b\rangle I\right) - \frac{\langle a,b\rangle}{14}\left(\frac{5}{6} cd - \frac{1}{7}\langle c,d\rangle I\right) \\
&\quad + ab \times cd - \frac{7\langle a,b\rangle\langle c,d\rangle}{6\cdot 14^2} I \\
&= \frac{5}{48\cdot 7}\left(\langle a,c\rangle\langle b,d\rangle + \langle a,d\rangle\langle b,c\rangle\right) I + \frac{1}{12}\langle a,b\rangle cd + \frac{1}{12}\langle c,d\rangle ab + ab \times cd \\
&= -\frac{1}{48}\left((a*c)(b*d) + (a*d)(b*c)\right) \\
&\quad + \frac{1}{12}\left(\langle a,c\rangle bd + \langle a,d\rangle bc + \langle b,c\rangle ad + \langle b,d\rangle ac + \langle a,b\rangle cd + \langle c,d\rangle ab\right). \qquad \blacksquare
\end{aligned}
$$

To end this section, we give a formula for τ under this isomorphism.

Proposition 3.3.9. *The bilinear form τ under the isomorphism σ is given by*

$$\tau(ab, cd) = \frac{1}{7\cdot 24}(5\langle a,c\rangle\langle b,d\rangle + 5\langle a,d\rangle\langle b,c\rangle + 2\langle a,b\rangle\langle c,d\rangle).$$

Proof. We compute τ by the definition:

$$
\begin{aligned}
\tau(ab, cd) &= \varepsilon(ab \diamond cd) \\
&= \varepsilon\left(\frac{1}{12}\left(\langle a,c\rangle bd + \langle a,d\rangle bc + \langle b,c\rangle ad + \langle b,d\rangle ac + \langle a,b\rangle cd + \langle c,d\rangle ab\right)\right. \\
&\qquad \left. -\frac{1}{48}\left((a*c)(b*d) + (a*d)(b*c)\right)\right) \\
&= \frac{1}{7\cdot 48}\left(8(\langle a,c\rangle\langle b,d\rangle + \langle a,d\rangle\langle b,c\rangle + \langle a,b\rangle\langle c,d\rangle)\right. \\
&\qquad \left. - (\langle a*c, b*d\rangle + \langle a*d, b*c\rangle)\right).
\end{aligned}
$$

As in the previous proof (by Lemma 3.2.19), we have $\langle a*c, b*d\rangle + \langle a*d, b*c\rangle = -2\langle a,c\rangle\langle b,d\rangle - 2\langle a,d\rangle\langle b,c\rangle + 4\langle c,d\rangle\langle a,b\rangle$. So we get

$$\tau(ab, cd) = \frac{1}{7\cdot 24}\left(5\langle a,c\rangle\langle b,d\rangle + 5\langle a,d\rangle\langle b,c\rangle + 2\langle a,b\rangle\langle c,d\rangle\right). \qquad \blacksquare$$

4 The compact real form of $A(\mathfrak{g}_2)$

Next, we will use the formulas from the previous chapter to describe one very specific case of *Galois descent.*

We will use the method described in [Kna02, Theorem 6.4] and explained for this specific situation in an unpublished document by Dr. Michiel van Couwenberghe. As a side result, we describe a Chevalley basis of $\mathfrak{g}_2$ in function of the standard derivations. At this point we will of course denote the octonions over the complex numbers $\mathbb{C}$ as $\mathbb{O}$, and denote $\mathbf{i} \in \mathbb{C}$ such that $\mathbf{i}^2 = -1$.

4.1 A description of Chevalley bases using standard derivations

To describe a Chevalley basis, we of course need to find a Cartan subalgebra. We will start with a combinatorial lemma.

Lemma 4.1.1. *Let $c \in W \subset \mathbb{O}$ be an imaginary octonion of norm 1, and $a_1, b_1, a_2, b_2, a_3, b_3 \in W$ imaginary octonions such that*

- *$a_i b_i = c$ for all $i = 1, 2, 3$,*
- *$c, a_1, b_1, a_2, b_2, a_3, b_3$ is an orthonormal basis of W,*
- *$a_1 a_2 = a_3$.*

Furthermore, define $d_{i,j} \in \{0, 1\}$ such that $a_i a_j = (-1)^{d_{i,j}} a_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Then we have the following identities:

$$d_{1,2} = 0, \ d_{1,3} = 1, \ \text{and } d_{2,3} = 0.$$

And

$$T := \langle D_{a_i,b_i} \mid i = 1, 2, 3 \rangle$$

is a Cartan subalgebra.

Proof. We calculate $[D_{a_1,b_1}, D_{a_2,b_2}]$. The other cases are completely analogous.

$$\begin{aligned} [D_{a_1,b_1}, D_{a_2,b_2}] &= D_{D_{a_1,b_1}(a_2),b_2} + D_{a_2,D_{a_1,b_1}(b_2)} \\ &= D_{-2b_2,b_2} + D_{a_2,2a_2} \\ &= 0 \end{aligned}$$

Thus T is nilpotent. As T is at least 2-dimensional, it has to be maximal, since any Cartan subalgebra of $\mathfrak{g}_2$ is 2-dimensional. ■

In the next proposition, we determine the weight vectors of this Cartan subalgebra.

Proposition 4.1.2. *Let $c, a_1, b_1, a_2, b_2, a_3, b_3$ be as in the previous lemma. Then $D_{c,a_i+\mathbf{i}b_i}, D_{c,a_i-\mathbf{i}b_i}$ are eigenvectors for the Cartan subalgebra T.*

Proof. Note that if $v_1, v_2 \in W$ are eigenvectors of $D_{a,b}$ with eigenvalues λ_1, λ_2 we have

$$\begin{aligned}[D_{a,b}, D_{v_1,v_2}] &= D_{D_{a,b}(v_1),v_2} + D_{v_1,D_{a,b}(v_2)} \\ &= \lambda_1 D_{v_1,v_2} + \lambda_2 D_{v_1,v_2}.\end{aligned}$$

In other words, in this way we can construct roots with respect to T. We determine the eigenvectors of D_{a_i,b_i}. First note that c is an eigenvector with eigenvalue 0, by. Next we check that $a_i \pm \mathbf{i}b_i$ is an eigenvector:

$$\begin{aligned}D_{a_i,b_i}(a_i \pm \mathbf{i}b_i) &= 4b_i \mp 4\mathbf{i}a_i \\ &= \mp 4\mathbf{i}(a_i \pm \mathbf{i}b_i).\end{aligned}$$

For $j \neq i$ we get

$$\begin{aligned}D_{a_i,b_i}(a_j \pm \mathbf{i}b_j) &= -2b_j \pm 2\mathbf{i}a_j \\ &= \pm 2\mathbf{i}(a_j \pm \mathbf{i}b_j).\end{aligned}$$

■

Now we need to analyze the combinatorial data we have gathered and rescale the vectors to identify the root system G_2 in this setting. We will use the following lemma:

Lemma 4.1.3. *For c, a_i, b_i, with $i \in \{1,2,3\}$ as in Lemma 4.1.1, we have (for $i, j \in \{1,2,3\}, i \neq j$ and $k, l \in \mathbb{N}$):*

- $c \cdot (a_i + (-1)^k \mathbf{i}b_i) = (-1)^{k+1}\mathbf{i}(a_i + (-1)^k \mathbf{i}b_i)$,
- $(a_i + (-1)^k \mathbf{i}b_i) \cdot (a_i + (-1)^{k+1}\mathbf{i}b_i) = -2 + (-1)^{k+1}2\mathbf{i}c$,
- $(a_i + (-1)^k \mathbf{i}b_i) \cdot (a_i + (-1)^k \mathbf{i}b_i) = 0$,
- $(a_i + (-1)^k \mathbf{i}b_i) \cdot (a_j + (-1)^l \mathbf{i}b_j) = 2\delta_{k,l}(-1)^{d_{i,j}}(a_k + (-1)^{k+1}\mathbf{i}b_k)$.

Here $\delta_{k,l}$ denotes the Dirac delta.

Proof. We only compute the last point, since the others do not involve technical details. In the last point we will need the Moufang identities 3.2.8. We have

$$\begin{aligned}(a_i + (-1)^k \mathbf{i}b_i) \cdot (a_j + (-1)^l \mathbf{i}b_j) &= a_i \cdot a_j + (-1)^l \mathbf{i}a_i \cdot (c \cdot a_j) \\ &\quad + (-1)^k \mathbf{i}(c \cdot a_i) \cdot a_j + (-1)^{k+l+1}(c \cdot a_i) \cdot (c \cdot a_j) \\ &= (-1)^{d_{i,j}} a_k + (-1)^{l+1}\mathbf{i}c \cdot (a_i \cdot a_j) \\ &\quad + (-1)^{k+1}\mathbf{i}c \cdot (a_i \cdot a_j) + (-1)^{k+l}(c \cdot a_i) \cdot (a_j \cdot c) \\ &= (-1)^{d_{i,j}} a_k + (-1)^{l+1+d_{i,j}}\mathbf{i}c \cdot a_k \\ &\quad + (-1)^{k+1+d_{i,j}}\mathbf{i}c \cdot a_k + (-1)^{k+l} c \cdot ((a_i \cdot a_j) \cdot c) \\ &= (1 + (-1)^{k+l})(-1)^{d_{i,j}}(a_k + (-1)^{k+1}\mathbf{i}b_k),\end{aligned}$$

by repeated use of the second point of Lemma 3.2.3 and Proposition 3.2.8. ■

Keeping in mind Proposition 3.2.26, we first compute some explicit derivations. The previous lemma will be of use.

Lemma 4.1.4. *Let c, a_i, b_i be as in Lemma 4.1.1. We have the following identities for $\{i,j,k\} = \{1,2,3\}$ and $l, m \in \mathbb{N}$:*

1.

$$D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^m\mathbf{i}b_j}\left(a_i+(-1)^{l+1}\mathbf{i}b_i\right)=\left(8+(-1)^{l+m+1}4\right)\left(a_j+(-1)^m\,\mathbf{i}b_j\right),$$

2.

$$D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^m\mathbf{i}b_j}\left(a_j+(-1)^{m+1}\mathbf{i}b_j\right)=-\left(8+(-1)^{l+m+1}4\right)\left(a_i+(-1)^l\,\mathbf{i}b_i\right),$$

3.

$$D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^m\mathbf{i}b_j}\left(a_k+(-1)^\gamma\mathbf{i}b_k\right)=0$$

unless $l=m=\gamma$, and then we have

$$D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^l\mathbf{i}b_j}\left(a_k+(-1)^l\mathbf{i}b_k\right)=(-1)^{d_{i,j}+l+1}8\mathbf{i}c.$$

Proof. 1. We handle the first identity using Corollary 3.2.29:

$$\begin{aligned}&D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^m\mathbf{i}b_j}\left(a_i+(-1)^{l+1}\mathbf{i}b_i\right)\\&=3\left\langle a_i+(-1)^l\mathbf{i}b_i,a_i+(-1)^{l+1}\mathbf{i}b_i\right\rangle\left(a_j+(-1)^m\,\mathbf{i}b_j\right)\\&\quad+\frac{1}{2}\left(a_i+(-1)^{l+1}\mathbf{i}b_i\right)*\left(\left(a_i+(-1)^l\mathbf{i}b_i\right)*\left(a_j+(-1)^m\mathbf{i}b_j\right)\right).\end{aligned}$$

In this last expression, we can use Lemma 3.2.19 to change the orders of multiplication. Then using Lemma 4.1.3, we get

$$\begin{aligned}&=12\left(a_j+(-1)^m\,\mathbf{i}b_j\right)\\&\quad+\frac{1}{2}\left(\left(a_i+(-1)^l\mathbf{i}b_i\right)*\left(a_i+(-1)^{l+1}\mathbf{i}b_i\right)\right)*\left(a_j+(-1)^m\mathbf{i}b_j\right)-4(a_j+(-1)^m\mathbf{i}b_j)\\&=12\left(a_j+(-1)^m\,\mathbf{i}b_j\right)\\&\quad+(-1)^{l+1}2\mathbf{i}c*\left(a_j+(-1)^m\mathbf{i}b_j\right)-4(a_j+(-1)^m\mathbf{i}b_j)\\&=8\left(a_j+(-1)^m\,\mathbf{i}b_j\right)\\&\quad+(-1)^{m+l+1}4\left(a_j+(-1)^m\mathbf{i}b_j\right)\\&=\left(8+(-1)^{l+m+1}4\right)\left(a_j+(-1)^m\,\mathbf{i}b_j\right).\end{aligned}$$

2. By (anti)symmetry, we also have

$$D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^m\mathbf{i}b_j}\left(a_j+(-1)^{m+1}\mathbf{i}b_j\right)=-\left(8+(-1)^{l+m+1}4\right)\left(a_i+(-1)^l\,\mathbf{i}b_i\right).$$

3. The third identity becomes (using Lemma 4.1.3)

$$\begin{aligned}&D_{a_i+(-1)^l\mathbf{i}b_i,a_j+(-1)^m\mathbf{i}b_j}\left(a_k+(-1)^\gamma\mathbf{i}b_k\right)\\&=\frac{1}{2}\left(a_k+(-1)^\gamma\mathbf{i}b_k\right)*\left(\left(a_i+(-1)^l\mathbf{i}b_i\right)*\left(a_j+(-1)^m\mathbf{i}b_j\right)\right)\\&=\delta_{l,m}(-1)^{d_{i,j}}2\left(a_k+(-1)^\gamma\mathbf{i}b_k\right)*\left(a_k+(-1)^{l+1}\mathbf{i}b_k\right).\end{aligned}$$

This is clearly only non-zero when $l=m=\gamma$ holds, by Lemma 4.1.3. And in that case, we get

$$\begin{aligned}&=(-1)^{d_{i,j}}2\left(a_k+(-1)^m\mathbf{i}b_k\right)*\left(a_k+(-1)^{m+1}\mathbf{i}b_k\right)\\&=(-1)^{m+1+d_{i,j}}8\mathbf{i}c.\end{aligned}$$

■

The last lemma we prove before tackling the the entire Chevalley basis, shows which roots are opposite.

Lemma 4.1.5. *For $i, j \in \{1,2,3\}$ with $i \neq j$ we have*

$$\begin{aligned}&[D_{a_i+(-1)^k \mathbf{i} b_i, a_j+(-1)^l \mathbf{i} b_j}, D_{a_i+(-1)^{k+1}\mathbf{i} b_i, a_j+(-1)^{l+1}\mathbf{i} b_j}] \\ &= 8\mathbf{i}\left(2+(-1)^{k+l+1}\right)\left((-1)^{l+1}D_{a_j,b_j}+(-1)^{k+1}D_{a_i,b_i}\right),\end{aligned}$$

which is an element of the Cartan subalgebra T.

Proof. Using the results of the previous lemma, together with Proposition 3.2.26, we get

$$\begin{aligned}&[D_{a_i+(-1)^k \mathbf{i} b_i, a_j+(-1)^l \mathbf{i} b_j}, D_{a_i+(-1)^{k+1}\mathbf{i} b_i, a_j+(-1)^{l+1}\mathbf{i} b_j}] \\ &= D_{D_{a_i+(-1)^k \mathbf{i} b_i, a_j+(-1)^l \mathbf{i} b_j}(a_i+(-1)^{k+1}\mathbf{i} b_i), a_j+(-1)^{l+1}\mathbf{i} b_j} \\ &\quad + D_{a_i+(-1)^{k+1}\mathbf{i} b_i, D_{a_i+(-1)^k \mathbf{i} b_i, a_j+(-1)^l \mathbf{i} b_j}(a_j+(-1)^{l+1}\mathbf{i} b_j)} \\ &= \left(8+(-1)^{k+l+1}4\right) D_{a_j+(-1)^l \mathbf{i} b_j, a_j+(-1)^{l+1}\mathbf{i} b_j} \\ &\quad - \left(8+(-1)^{k+l+1}4\right) D_{a_i+(-1)^{k+1}\mathbf{i} b_i, a_i+(-1)^k \mathbf{i} b_i} \\ &= \left(8+(-1)^{k+l+1}4\right)\left(2(-1)^{l+1}\mathbf{i} D_{a_j,b_j}+2(-1)^{k+1}\mathbf{i} D_{a_i,b_i}\right).\end{aligned}$$ ■

Note that the previous lemma suggests something about the "size" of our roots. It depends on the signs k and l. The roots where k and l differ will precisely be our long roots, and when they are the same, the root vector will correspond to a short root.

At this point, we are able to formulate the precise form of a Chevalley basis.

Proposition 4.1.6. *A Chevalley basis with respect to the Cartan subalgebra T from Lemma 4.1.1 is given by Figure 4.1, where the label of each root γ corresponds to the root vector X_γ. The roots H_γ in the maximal torus are given in Table 4.1.*

Proof. Recall that the root system G_2 is given by

$$\{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$$

where α is a short root and β is a long root, with $\alpha(\beta) = -3$ and $\beta(\alpha) = -1$.

We can use the previous computations to verify all equations in the definition of a Chevalley basis. This is a tedious exercise, but quite straightforward. There are only a couple cases where we have to use Lemma 3.2.27, as well as for the computations in Table 4.1. To illustrate these cases, we compute the following:

$$\begin{aligned}&\left[\frac{1}{4}(D_{a_1+\mathbf{i}b_1, a_2+\mathbf{i}b_2}), \frac{1}{4}(D_{a_1+\mathbf{i}b_1, a_3+\mathbf{i}b_3})\right] \\ &= \frac{1}{16}\left(D_{D_{a_1+\mathbf{i}b_1,a_2+\mathbf{i}b_2}(a_1+\mathbf{i}b_1), a_3+\mathbf{i}b_3} + D_{a_1+\mathbf{i}b_1, D_{a_1+\mathbf{i}b_1,a_2+\mathbf{i}b_2}(a_3+\mathbf{i}b_3)}\right) \\ &= \frac{1}{16}\left(D_{a_1+\mathbf{i}b_1, -8\mathbf{i}c}\right)\end{aligned}$$

by Lemma 4.1.4. Now, we prove this is equal to $-\frac{2}{4}D_{a_2-\mathbf{i}b_2, a_3-\mathbf{i}b_3}$:

$$
\begin{aligned}
-\frac{1}{4}D_{a_2-\mathbf{i}b_2,a_3-\mathbf{i}b_3} &= -\frac{1}{4}D_{(e-\mathbf{i}c)a_2,a_3-\mathbf{i}b_3} \\
&= \frac{1}{4}(D_{(a_3-\mathbf{i}b_3)(e-\mathbf{i}c),a_2} + D_{a_2(a_3-\mathbf{i}b_3),(e-\mathbf{i}c)}) \\
&= \frac{1}{4}(0 + D_{a_1+\mathbf{i}b_1,-\mathbf{i}c}).
\end{aligned}
$$

The other cases are analogous. ■

H_α	$H_{\alpha+\beta}$	$H_{2\alpha+\beta}$	H_β	$H_{3\alpha+\beta}$	$H_{3\alpha+2\beta}$
$\frac{1}{2\mathbf{i}}D_{a_3,b_3}$	$\frac{1}{2\mathbf{i}}D_{a_2,b_2}$	$-\frac{1}{2\mathbf{i}}D_{a_1,b_1}$	$\frac{1}{6\mathbf{i}}(D_{a_3,b_3} - D_{a_2,b_2})$	$\frac{1}{6\mathbf{i}}(D_{a_1,b_1} - D_{a_3,b_3})$	$\frac{1}{6\mathbf{i}}(D_{a_1,b_1} - D_{a_2,b_2})$

Table 4.1: The roots in the Cartan subalgebra T from Lemma 4.1.1.

4.2 Describing the compact real form

Suppose we have a split complex Lie algebra $\mathfrak{g}$ with Chevalley basis

$$\{H_\alpha, X_\alpha, X_{-\alpha} \mid \alpha \in \Phi\}.$$

Then the $\mathbb{R}$-span (denoted $\mathfrak{g}'$) of the elements

$$
\begin{aligned}
h_\alpha &:= \mathbf{i}H_\alpha \\
x_\alpha &:= X_\alpha - X_{-\alpha} \\
y_\alpha &:= \mathbf{i}(X_\alpha + X_{-\alpha})
\end{aligned}
$$

is a compact real form of $\mathfrak{g}$ (See [Kna02, Theorem 6.4]).

Proposition 4.2.1. *The set*

$$A' := \{S(A_1A_2) \mid A_1, A_2 \in \mathfrak{g}'\}$$

is a subalgebra of $A(\mathfrak{g}_2)$, and it is closed under the action of $\mathfrak{g}'$.

Proof. This is immediately clear for the action of $\mathfrak{g}'$. The fact that this $\mathbb{R}$-span is also closed under the algebra product follows from the fact that $\mathfrak{g}'$ is closed under the Lie brackets, and the definition of the algebra product. ■

Proposition 4.2.2. $A' \otimes \mathbb{C} = A(\mathfrak{g}_2)$.

We will now determine what this A' looks like, where we use a Chevalley basis as in the previous section, with

$$c = e_1, a_1 = e_2, b_1 = e_3, a_2 = e_4, b_2 = e_5, a_3 = e_6, b_3 = -e_7.$$

Proposition 4.2.3. *Under the identification of Proposition 3.3.8, we have that*

$$A' = \mathrm{S}^2\,\mathbb{R}\langle e_i \mid i = 1, \dots, 7\rangle,$$

with multiplication inherited from $A(\mathfrak{g}_2)$.

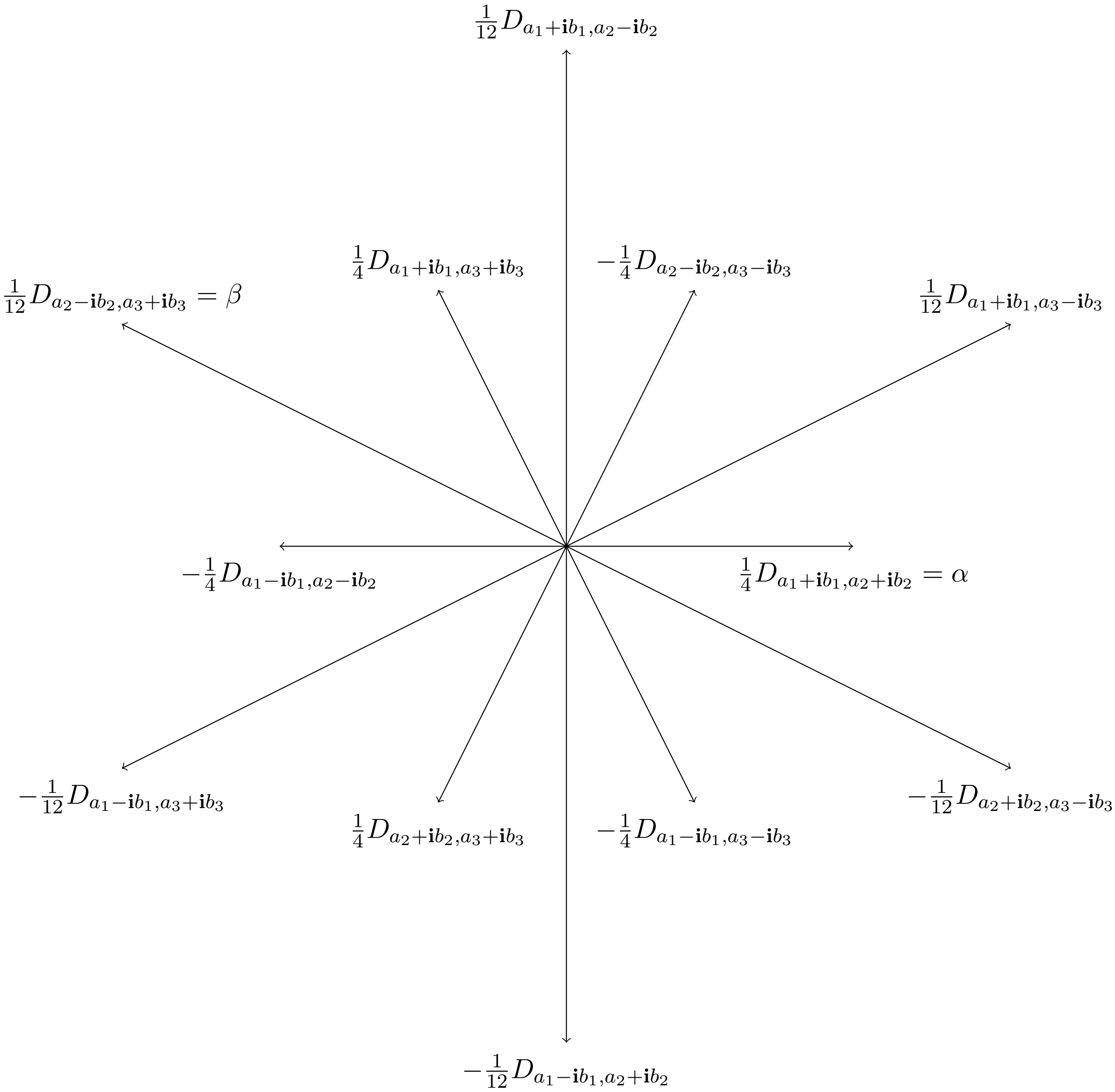


Figure 4.1: The root vectors of G_2 in this setting.

Proof. The recipe given above gives us the following derivations in this setting:

h_α	$h_{\alpha+\beta}$	$h_{2\alpha+\beta}$
$-\frac{1}{2}D_{e_6,e_7}$	$\frac{1}{2}D_{e_4,e_5}$	$-\frac{1}{2}D_{e_2,e_3}$
h_β	$h_{3\alpha+\beta}$	$h_{3\alpha+2\beta}$
$-\frac{1}{6}\left(D_{e_4,e_5}+D_{e_6,e_7}\right)$	$\frac{1}{6}\left(D_{e_2,e_3}+D_{e_6,e_7}\right)$	$\frac{1}{6}\left(D_{e_2,e_3}-D_{e_4,e_5}\right)$
x_α	$x_{\alpha+\beta}$	$x_{2\alpha+\beta}$
$\frac{1}{2}\left(D_{e_2,e_4}-D_{e_3,e_5}\right)$	$\frac{1}{2}\left(D_{e_2,e_6}+D_{e_3,e_7}\right)$	$-\frac{1}{2}\left(D_{e_4,e_6}+D_{e_5,e_7}\right)$
x_β	$x_{3\alpha+\beta}$	$x_{3\alpha+2\beta}$
$\frac{1}{6}\left(D_{e_4,e_6}-D_{e_5,e_7}\right)$	$\frac{1}{6}\left(D_{e_2,e_6}-D_{e_3,e_7}\right)$	$\frac{1}{6}\left(D_{e_2,e_4}+D_{e_3,e_5}\right)$
y_α	$y_{\alpha+\beta}$	$y_{2\alpha+\beta}$
$-\frac{1}{2}\left(D_{e_2,e_5}+D_{e_3,e_4}\right)$	$\frac{1}{2}\left(D_{e_2,e_7}-D_{e_3,e_6}\right)$	$\frac{1}{2}\left(D_{e_4,e_7}-D_{e_5,e_6}\right)$
y_β	$y_{3\alpha+\beta}$	$y_{3\alpha+2\beta}$
$\frac{1}{6}\left(D_{e_4,e_7}+D_{e_5,e_6}\right)$	$-\frac{1}{6}\left(D_{e_3,e_6}+D_{e_2,e_7}\right)$	$\frac{1}{6}\left(D_{e_2,e_5}-D_{e_3,e_4}\right)$

From this table we see that

$$\mathfrak{g}' = \left\{ \sum_i D_{a_i,b_i} \,\middle|\, a_i, b_i \in \mathbb{R}\langle e_2, e_3, e_4, e_5, e_6, e_7\rangle \right\}.$$

In fact, by Lemma 3.2.27, a_i, b_i can be in the $\mathbb{R}$-span of $e_1, e_2, e_3, e_4, e_5, e_6, e_7$. For example, we prove that $D_{e_1,e_2} \in A'$. The other cases are completely analogous.

$$\begin{aligned} D_{e_1,e_2} &= D_{e_1,e_4e_6} \\ &= -D_{e_6,e_1e_4} - D_{e_4,e_6e_1} \\ &= -D_{e_6,e_5} - D_{e_4,e_7} \in \mathfrak{g}'. \end{aligned}$$

In other words, we have

$$\mathfrak{g}' = \left\{ \sum_i D_{a_i,b_i} \,\middle|\, a_i, b_i \in \mathbb{R}\langle e_1, e_2, e_3, e_4, e_5, e_6, e_7\rangle \right\}.$$

By the formulas derived in the previous section, it is then easy to see that

$$A' = \mathrm{S}^2\,\mathbb{R}\langle e_i \mid i = 1, \ldots, 7\rangle. \qquad \blacksquare$$

Remark 4.2.4. Note that A' in this case is precisely the symmetric square of the compact real form of the octonions. This is exactly what we would hope to get, also by the previous chapter.

5 Determining the automorphism group

In [CG21, Remark 9.2], it was noted we do not know the automorphism group yet of the algebra $A(\mathfrak{g}_2)$. We aim to resolve this issue, at least over the complex numbers (though similarly to previous results, it should hold more generally).

For that we expand on an observation given in [GG15], and further explained to me by Skip Garibaldi.

At this point it is inevitable to use the theory of algebraic groups, but there are many standard textbooks like [Hum75], or more recently [Mil17]. We work over $k = \mathbb{C}$, so we can conflate the algebraic group with the group of k-points $G(k)$.

Proposition 5.1.1. *The bilinear form τ from Definition 2.5.1 gives rise to a non-zero symmetric trilinear form given by*

$$\begin{aligned} T\colon A(\mathfrak{g}_2)^3 &\to k \\ (X, Y, Z) &\mapsto \tau(X \diamond Y, Z). \end{aligned}$$

It remains non-zero when restricting to $\ker \varepsilon = V$.

Proof. The trilinear form is symmetric because $\diamond$ is commutative and τ is associative.

One can verify that T is non-zero on $V = \ker \varepsilon$, since

$$T(e_1e_3, e_1e_2, e_2e_3) = \frac{5}{504}$$

by using the formula

$$\tau(ab, cd) = \frac{1}{7 \cdot 24}(5\langle a, c\rangle\langle b, d\rangle + 5\langle a, d\rangle\langle b, c\rangle + 2\langle a, b\rangle\langle c, d\rangle)$$

for all imaginary octonions a, b, c, d. ■

We will need the following fact:

Proposition 5.1.2. *Any automorphism of G_2 is inner.*

Proof. This follows from [Hum75, Theorem 27.4] and the fact that the Dynkin diagram of G_2 has no symmetries. ■

Before we go on, we also mention that Schur's Lemma also holds in this context.

Lemma 5.1.3 (Schur's Lemma). *Suppose a group G acts irreducibly on a representation V over an algebraically closed field k, and $\phi \in \mathrm{GL}(V)$ commutes with every element of G. Then ϕ acts as $\phi(w) = aw$ on all $w \in V$, for a certain scalar $a \in k$.*

Proof. Since we work over an algebraically closed field, we can find an eigenvector $v \in V$ of ϕ with eigenvalue a. But then so is gv for all g in G_2, since

$$\phi gv = g\phi v = agv$$

holds. But since G acts irreducibly on V, we have $V = \langle gv \mid g \in G_2 \rangle$. Thus $\phi(w) = aw$ for any $w \in V$. ■

A very instrumental article for this argument is [Sei87]. This paper sums up all the possibilities for connected groups that contain G_2.

Proposition 5.1.4. *Suppose $G_2 < H \leq \mathrm{SL}(V)$, and moreover that H is connected. Then one of the following occurs*

- $H = \mathrm{SL}(V)$,
- $H = \mathrm{SO}(V)$,
- *H is of type B_3 and acts on V as the representation with highest weight $2\omega_1$,*
- *H is of type E_6 and acts on V as the representation with highest weight ω_6.*

Proof. By [Sei87, Theorem 2], all connected overgroups of G_2 in $\mathrm{SL}(V)$ are listed in [Sei87, Table 1], or are of the form $\mathrm{SO}(V)$ or $\mathrm{SL}(V)$. The group $\mathrm{Sp}(V)$ does not exist, as V is odd-dimensional. Since we work over characteristic 0, we have the cases in the statement of the theorem. ■

Notation 5.1.5. We will denote the identity component of the stabilizer of the nondegenerate bilinear form $\tau|_V$ by $B \cong \mathrm{SO}(V)$.

Proposition 5.1.6. *The group G_2 is contained in exactly one copy B' of SO_7 in B.*

Proof. 1. Existence:

As we realised $A(\mathfrak{g}_2) = \mathrm{S}^2 W$ as the symmetric square of the purely imaginary octonions, we can consider the action of $B' = \mathrm{SO}_7$ stabilizing the bilinear form on the purely imaginary octonions. Then this stabilizes in particular $\tau|_V$ by the formula derived in Proposition 3.3.9.

2. Uniqueness:

Suppose $A \cong \mathrm{SO}_7$ is another overgroup of G_2 in B. Then it also has to stabilize the bilinear form $\tau|_V$. By Proposition 5.1.4, we know what the action of A on V should be. We will construct this representation now.

Take the 7-dimensional representation V' of A (denote the associated bilinear form by $\langle \cdot, \cdot \rangle'$). Then construct the symmetric square $\mathrm{S}^2 V'$ and restrict to the subspace

$$V_A = \left\{ \sum_i \lambda_i a_i b_i \in \mathrm{S}^2 V' \;\middle|\; \sum_i \langle a_i, b_i \rangle' = 0 \right\}.$$

Then this is the representation of B_3 with highest weight $2\omega_1$ (see also Remark 1.4.14). This representation comes equipped with the bilinear form

$$\tau_A(ab, cd) = \frac{1}{7 \cdot 24}(5\langle a, c\rangle' \langle b, d\rangle' + 5\langle a, d\rangle' \langle b, c\rangle').$$

So we have in fact $V_A \cong V$. Moreover, since both SO_7 copies only stabilize a 1-dimensional space of bilinear forms, the isomorphism sends τ_A to a scalar multiple of τ. Since we work over an algebraically closed field, we can assume the isomorphism sends τ_A to τ. Call this isomorphism φ.

Note that we can choose ϕ such that $A = \varphi \, \mathrm{SO}(V')\varphi^{-1}$, where $\mathrm{SO}(V')$ stands for the "natural" action on V_A.

Now pick an orthonormal basis $a_1, \ldots, a_7 \in V'$ and an orthonormal basis $b_1, \ldots, b_7 \in W$. Then we have an element $\rho \in \mathrm{O}(V)$ such that

$$\begin{aligned} \rho(\phi(a_1a_1 - a_ia_i)) &= b_1b_1 - b_ib_i, \\ \rho(\phi(a_ia_j)) &= b_ib_j. \end{aligned}$$

But then we have $B' = \rho A \rho^{-1}$, and conjugation by ρ stabilizes $B \cong \mathrm{SO}(V)$, as $\mathrm{SO}(V)$ is normal in $\mathrm{O}(V)$.

Now it could be that conjugation by ρ does not send G_2 back into itself. However, any two copies of G_2 in B' are also conjugate. By [SV00, Theorem 1.7.1], any similarity between composition algebras C, C' (a linear bijection that is a similarity between the bilinear forms) gives rise to an isomorphism between the composition algebras C, C'. Let C be the octonion algebra associated to G_2, and C' the octonion algebra associated to $\rho G_2 \rho^{-1}$. The associated bilinear forms to C and C' have to be equal up to a scalar, since B' only stabilizes a one dimensional space of bilinear forms on this representation (see Remark 1.4.14). Then the identity of the underlying vector space $W \oplus k$ is a similarity $C \to C'$. Thus we have an isomorphism $\psi : C \to C'$. Then by definition, $\psi^{-1}\rho G_2 \rho^{-1}\psi = G_2$.

Since ψ is in particular an isometry of the two associated bilinear forms, ψ is also a similarity of the bilinear form $\langle\cdot,\cdot\rangle$. Denote the multiplier of ψ by μ. Then, for any $r \in B'$ and $a, b \in W$ we have

$$\langle \psi^{-1} r \psi(a), \psi^{-1} r \psi(b)\rangle = \mu^{-1}\langle \psi(a), \psi(b)\rangle = \langle a, b\rangle.$$

Since B' is connected, it then follows that $\psi^{-1}B'\psi = B'$.

The automorphism ψ also acts on V, by sending ab to $\psi(a)\psi(b)$. We will also denote this map as ψ.

So in fact we have an element $\theta \in \mathrm{GL}(V)$ such that conjugation by θ sends A to B' and stabilizes G_2.

However, by Proposition 5.1.2, this means that there is an $h \in G_2$ such that for all $g \in G_2$

$$\theta g \theta^{-1} = hgh^{-1}.$$

By replacing θ with $h^{-1}\theta$, we can assume θ commutes with G_2. By Schur's Lemma then, θ acts as a scalar multiplication on V, since G_2 acts irreducibly on V. But then conjugation by θ is simply the identity isomorphism, so we conclude that $A = B'$. ■

Corollary 5.1.7. *Any connected group of type B_3 containing G_2 that stabilizes $\tau|_V$ is contained in B'.*

Proof. As in the previous proof, we can construct for a group A of type B_3 the representation V_A of highest weight $2\omega_1$, with an associated nonzero bilinear form $\langle\cdot,\cdot\rangle'$. This form is nondegenerate, since its radical would be an A-invariant subspace. By the formula of the constructed bilinear form τ_A, and the fact that $\tau_A = \lambda\tau$ for a certain $\lambda \in \mathbb{C}^*$, we have a copy of O_7 that also stabilizes $\tau|_V$ and contains A. As A is connected, it is contained in the identity component SO_7. As SO_7 is connected, it is contained in $B = \mathrm{SO}(V)$. Thus we know this copy of SO_7 has to be B' by the previous proposition. ■

Because a group of type B_3 is merely isogenous to SO_7, we include the following corollary as well.

Corollary 5.1.8. *Any group A of type B_3 contained in B' does not stabilize the symmetric trilinear form $T|_V$.*

Proof. The group A stabilizes a 1-dimensional space of symmetric trilinear forms by Remark 1.4.14. We can construct this trilinear form as follows:

First we define a symmetric trilinear form on $W \otimes W$ by

$$\begin{aligned} (W \otimes W)^3 &\to k \\ (a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3) &\mapsto \sum_{\sigma \in \mathcal{S}_3} \langle a_{\sigma(1)}, a_{\sigma(2)} \rangle \langle b_{\sigma(1)}, a_{\sigma(3)} \rangle \langle b_{\sigma(2)}, b_{\sigma(3)} \rangle. \end{aligned}$$

By composing this map with the inclusion

$$\begin{aligned} \mathrm{S}^2 V' &\to V' \otimes V' \\ ab &\mapsto \frac{a \otimes b + b \otimes a}{2} \end{aligned}$$

we get a symmetric trilinear form on $\mathrm{S}^2 W$. After restricting to V, we get an A-invariant symmetric trilinear form T'. This trilinear form is also non-zero, since we have

$$T'(e_1e_3, e_1e_2, e_3e_2) = 6.$$

However, T and T' are linearly independent. One can for example see that

$$\begin{aligned} &T(e_1e_3, e_1e_2, e_2e_3) = \frac{5}{504}, \\ &T\left(e_1e_1 - \frac{1}{2}(e_2e_2 + e_3e_3), e_1e_1 - \frac{1}{2}(e_2e_2 + e_3e_3), e_1e_1 - \frac{1}{2}(e_2e_2 + e_3e_3)\right) = \frac{5}{42}, \\ &T'\left(e_1e_1 - \frac{1}{2}(e_2e_2 + e_3e_3), e_1e_1 - \frac{1}{2}(e_2e_2 + e_3e_3), e_1e_1 - \frac{1}{2}(e_2e_2 + e_3e_3)\right) = 36. \end{aligned}$$

In other words, A does not stabilize T'. ■

This next argument will be needed to generalize the results of Proposition 5.1.4.

Proposition 5.1.9. *Suppose G is a closed connected subgroup of $\mathrm{GL}(V)$ such that*

1. *G acts irreducibly on V,*
2. *G is semisimple,*
3. *there is a nonzero homogenous $f \in k[V]^G$ so that G is the identity component of the stabilizer of f in $\mathrm{SL}(V)$.*

Then G is also the identity component of the stabilizer of f in $\mathrm{GL}(V)$.

Proof. This proof is due to Skip Garibaldi.

Put G' for the subgroup of $x \in \mathrm{SL}(V)$ so that there exists a $c_x \in k^*$ such that $f(xv) = c_x f(v)$ for all $v \in V$.

Claim A. G' has identity component G.

Evidently G' contains G, so it too acts irreducibly on V. In particular, it does not fix any vector in V, so it has trivial unipotent radical. That is, the identity component G'° of G' is reductive. Its central torus commutes with G' and acts nontrivially on V, so it must be contained in the scalar matrices. Since G' is in $\mathrm{SL}(V)$, the central torus is trivial. Thus G'° is semisimple.

The map sending x to c_x is a homomorphism to the multiplicative group $G' \to G_m$. But G'° is semisimple, so it has no nontrivial maps to G_m. That is, $c_x = 1$ for $x \in G'^{\circ}$. That is, G'° is contained in G. We conclude they are equal.

Now put $\mathbb{G}$ for the stabilizer of f in $\mathrm{GL}(V)$.

Claim B. $\mathbb{G}$ has identity component G.

There is a surjection $G_m \times \mathrm{SL}(V) \to \mathrm{GL}(V)$, defined by sending G_m to scalars in $\mathrm{GL}(V)$. Denote $\mathbb{G}'$ to be the inverse image of $\mathbb{G}$ under this map.

It has to consist of pairs (x, g) where g is in the group G'. Conversely, for any $g \in G'$ there are finitely many choices for $x \in k^*$ such that $xg \in \mathbb{G}$, because x has to satisfy the polynomial equation $x^{\deg f} - c_g$. This implies the group $\{(1, g) \mid g \in G\}$ has finite index in $\mathbb{G}'$. Namely, its index is at most $(\deg f)[G' : G]$. By [Hum75, Proposition 7.3], it then follows G is the identity component of $\mathbb{G}'$. By [Hum75, Proposition 7.4.B], we also have that G is the identity component of $\mathbb{G}$, concluding the proof. ∎

Corollary 5.1.10. *The group G_2 is normal in* $\mathrm{Aut}(A(\mathfrak{g}_2))$.

Proof. Let us look at the stabilizer of the symmetric trilinear form $T|_V$. $\mathrm{Aut}(A(\mathfrak{g}_2))$ stabilizes T, since it stabilizes the counit ε by [CG21, Example A.6]. The group G_2 is contained in the identity component of $T|_V$, since it is connected.

By Proposition 5.1.4, the identity component G_0 of the stabilizer of the symmetric trilinear form $T|_V$ in $\mathrm{SL}(V)$ is either of type E_6, B_3 or G_2. The options $H = \mathrm{SL}(V)$ and $H = \mathrm{SO}(V)$ are impossible, since these groups do not stabilize a symmetric trilinear form on their standard representation.

In either case, the identity component is semisimple, and acts irreducibly on V. Moreover, a symmetric trilinear form is equivalent to a cubic form in characteristic 0, and their stabilizers are equal. Thus Proposition 5.1.9 applies, and the identity component G_0 of the stabilizer in $\mathrm{SL}(V)$ is equal to the identity component of the stabilizer in $\mathrm{GL}(V)$.

We claim the intersection of $\mathrm{Aut}(A(\mathfrak{g}_2))$ with G_0 has to be G_2. In case G_0 has type E_6, we know that the representation with highest weight ω_6 does not stabilize a symmetric bilinear form, and thus does not stabilize $\tau|_V$. In case G_0 has type B_3, we also know it cannot stabilize the bilinear form $\tau|_V$, by Proposition 5.1.7 and Corollary 5.1.8. In case G_0 has type G_2 no additional argument is needed.

In any case, by [CG21, Example A.6], any automorphism of $A(\mathfrak{g}_2)$ has to stabilize $\tau|_V$. So the intersection of $\mathrm{Aut}(A(\mathfrak{g}_2))$ with G_0 is the group G_2 by the previous paragraph and Proposition 5.1.4. As G_0 is normal in the stabilizer of $T|_V$, this implies that G_2 is normal in $\mathrm{Aut}(A(\mathfrak{g}_2))$. ∎

Theorem 5.1.11. *We have*

$$\mathrm{Aut}(A(\mathfrak{g}_2)) = G_2.$$

Proof. Let $\theta \in \mathrm{Aut}(A(\mathfrak{g}_2)) \setminus G_2$. Then, because of Corollary 5.1.10, conjugation by θ is an automorphism of G_2. By Proposition 5.1.2, this means that there is an $h \in G_2$ such that for all $g \in G_2$

$$\theta g \theta^{-1} = hgh^{-1}$$

holds. In other words, $h^{-1}\theta$ commutes with G_2.

But if $\phi = h^{-1}\theta$ commutes with G_2, this also holds after resticting to $V = V(2\omega_1)$. This means ϕ acts as a scalar a on V by Schur's Lemma.

Then the image of e_1e_1 is

$$\begin{aligned}\phi(e_1e_1) &= \phi(e_1e_1 - \frac{\sum_{i=1}^7 e_ie_i}{7} + \frac{\sum_{i=1}^7 e_ie_i}{7}) \\ &= ae_1e_1 + (1-a)\frac{\sum_{i=1}^7 e_ie_i}{7}.\end{aligned}$$

Since e_1e_1 is an idempotent, its image under an automorphism of $A(\mathfrak{g}_2)$ is also an idempotent, i.e.

$$\begin{aligned}&(ae_1e_1 + (1-a)\frac{\sum_{i=1}^7 e_ie_i}{7}) \diamond (ae_1e_1 + (1-a)\frac{\sum_{i=1}^7 e_ie_i}{7}) \\ &= a^2e_1e_1 + \frac{4(1-a)a}{7}e_1e_1 + (1-a)^2\frac{\sum_{i=1}^7 e_ie_i}{7} \\ &= ae_1e_1 + (1-a)\frac{\sum_{i=1}^7 e_ie_i}{7}\end{aligned}$$

should hold. By comparing coefficients of e_2e_2, we get that $(1-a)^2 = (1-a)$, i.e. $a = 1$ or 0. Since ϕ is a bijection in particular, a cannot be 0. Thus $a = 1$ and ϕ is the identity. This implies $\theta = h$, which contradicts our choice of θ. ■

A Summary (in Dutch)

Deze thesis gaat over representaties van algebraïsche groepen. Algebraïsche groepen hebben een redelijk abstracte definitie, die redelijk wat technicaliteiten omvat, maar intuïtief kunnen we ze omschrijven als matrixgroepen gedefinieerd aan de hand van polynomiale vergelijkingen. Om inzicht in de structuur van deze algebraïsche groepen te krijgen, kijken we naar representaties van deze groepen; dit zijn objecten (meetkundig of algebraïsch) die de algebraïsche groepen als symmetrieën hebben.

In deze thesis kijken we specifiek naar een familie niet-associatieve algebra's die representaties vormen van zogenaamde *exceptionele algebraïsche groepen*. Deze algebra's werden onafhankelijk geïntroduceerd in [DMVC21] en [CG21].

Om te beginnen geven we een een introductie en verloop van dit proefschrift. In het eerste hoofdstuk bespreken we dan voorafgaande concepten zoals wortelsystemen, Lie algebra's, en een stukje representatietheorie. De hoop is dat we hiermee de lezer voldoende basisconcepten meegeven om de constructie van de algebra's uit [CG21] te doorgronden.

In het tweede hoofdstuk gaan we dan weer dieper in op de constructie van de niet-associatieve algebra's gegeven door [CG21]. Het hoofdstuk start met een gedetailleerde beschrijving van de constructie. Daarna geven we dan ook de hoofdresultaten van dit artikel mee, meestal zonder bewijs.

Vanaf het derde hoofdstuk breiden we uit op de resultaten gegeven in [CG21] voor het specifieke geval G_2. Het is al lang geweten dat de groep G_2 op natuurlijke wijze voorkomt als de symmetriegroep van de *octonionen*, een algebra die ontstaat als natuurlijke veralgemening van de complexe getallen en de quaternionen. In het derde hoofdstuk geven we een expliciete beschrijving van deze nieuwe algebra's, in termen van de octonionen.

Stelling (Hoofdresultaat 1). *De algebra $A(\mathfrak{g}_2)$ is isomorf met het symmetrisch kwadraat van de* zuiver imaginaire *octonionen* $\mathrm{S}^2 W$*, met vermenigvuldiging gegeven door*

$$\begin{aligned} ab \star cd = {} & \frac{1}{12}\left(\langle a,c\rangle bd + \langle a,d\rangle bc + \langle b,c\rangle ad + \langle b,d\rangle ac + \langle a,b\rangle cd + \langle c,d\rangle ab\right) \\ & - \frac{1}{48}\left((a*c)(b*d) + (a*d)(b*c)\right). \end{aligned}$$

Hierbij staat $*$ voor het zogenaamd *Maltsev product* op de zuiver imaginaire octonionen, een product afgeleid van het gewone octonionenproduct

Het vierde hoofdstuk gebruikt deze beschrijving om de methode van *Galois descent* toe te passen om nieuwe algebra's te construeren. Dit hoofdstuk is eerder het uitwerken van een voorbeeld dan een theorie uitwerken, en het voorbeeld lijkt te wijzen op een zeer eenvoudige beschrijving in termen van Galois descent op de corresponderende octonionalgebra's. We hebben het volgende hoofdresultaat:

Stelling (Hoofdresultaat 2). *De compacte reële vorm van de algebra $A(\mathfrak{g}_2)$ wordt gegeven door het symmetrisch kwadraat van de compacte reële vorm van de octonion, met vermenigvuldiging zoals in Hoofdresultaat 1.*

Het vijfde en laatste hoofdstuk gaat over de symmetriegroep van de algebra van type G_2. In [CG21] werd bewezen dat voor 2 van de 5 exceptionele types (namelijk F_4 en E_8) de symmetriegroep van de geconstrueerde algebra gelijk is aan de groep van hetzelfde type. In dit hoofdstuk breiden we dit resultaat uit, zodat het ook geldt voor type G_2. Om dit resultaat te bereiken, maken we ook gebruik van de expliciete beschrijving gevonden in hoofdstuk drie.

Stelling (Hoofdresultaat 3). *De automorfismengroep van de algebra $A(\mathfrak{g}_2)$ is exact de adjuncte ("adjoint") groep van type G_2.*

Bibliography

[Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[Bou05] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[Car89] Roger W. Carter. *Simple groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.

[CdM96] Arjeh M. Cohen and Ronald de Man. Computational evidence for Deligne's conjecture regarding exceptional Lie groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(5):427–432, 1996.

[CG21] Maurice Chayet and Skip Garibaldi. A class of continuous non-associative algebras arising from algebraic groups including e8. *Forum Math. Sigma*, 9:Paper No. e6, 2021.

[DMR17] Tom De Medts and Felix Rehren. Jordan algebras and 3-transposition groups. *J. Algebra*, 478:318–340, 2017.

[DMVC21] Tom De Medts and Michiel Van Couwenberghe. Non-associative frobenius algebras for simply laced chevalley groups. *Trans. Amer. Math. Soc.*, page to appear, 2021.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[GG15] Skip Garibaldi and Robert M. Guralnick. Simple groups stabilizing polynomials. *Forum Math. Pi*, 3:e3, 41, 2015.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[Mil17] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.

[PS08] Alexander Premet and Helmut Strade. Simple lie algebras of small characteristic vi. completion of the classification. *Journal of Algebra*, 320(9):3559 – 3604, 2008.

[Sch95] Richard D. Schafer. *An introduction to nonassociative algebras.* Dover Publications, Inc., New York, 1995. Corrected reprint of the 1966 original.

[Sei87] Gary M. Seitz. The maximal subgroups of classical algebraic groups. *Mem. Amer. Math. Soc.*, 67(365):iv+286, 1987.

[SS70] T. A. Springer and R. Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 167–266. Springer, Berlin, 1970.

[SV00] Tonny A. Springer and Ferdinand D. Veldkamp. *Octonions, Jordan algebras and exceptional groups.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.